ELEMENTARY OPHTHALMIC OPTICS

FREELAND · FERGUS

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BY

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TO

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PREFACE

The aim of this book is to set forth, in a clear and simple manner, those portions of Physical and Geometrical Optics which I consider essential to the medical student beginning his ophthalmic studies. No one recognizes more fully than I do that the proper province of the ophthalmic surgeon is to treat diseases of the eye; but in order that he may do so with success it is necessary that he should have a competent knowledge of the organ of vision from a physical stand-point.

It is hoped that this small manual will form a suitable and easy introduction to such knowledge. The subject of Physiological Optics is not discussed, since that branch of ophthalmic study is, as a rule, sufficiently explained in text-books dealing with diseases of the eye. My endeavour has been to give in one volume all the information necessary for the beginner, and therefore I have placed in the introduction those parts of plane trigonometry which he requires.

The works of Landolt, Tscherning, Glazebrook, Preston, and others have been freely consulted, and have been of much service.

Acknowledgment of indebtedness is due to Dr. W. Inglis Pollock, who did the work of amanuensis, to Principal M'Lean of the Paisley Technical College, and to Professor Peter Bennett, for much assistance in reading proofs.

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CONTENTS

ELEMENTARY OPHTHALMIC OPTICS

INTRODUCTION

The pages which follow are an attempt to place before the student of Ophthalmology a clear account of those parts of physical and geometrical optics which are of use to him in his studies. In a text-book which is intended to contain only such elements of the subject as are required by the every-day student, the information given can at best be but rudimentary.

As a preliminary, it may be well to define here those functions of an angle which are met with even in the most elementary investigations.

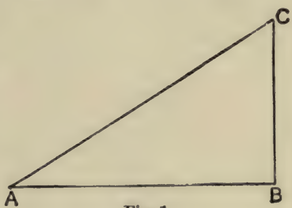


Fig. 1

Let A B C be a right-angled triangle, of which the angle at B is the right angle. The sine of the angle at A is defined to be the ratio of the length of the right line B C to that of the right line A C. In like manner the cosine of A is the ratio of A B to A C, and the tangent of the angle at A is the ratio of B C to A B.

These three ratios or functions are the only ones which we require for our present work, although other three functions of A are in common use, viz., the secant, the ratio of A C to A B; the cosecant, the ratio of A C to B C; and the cotangent, the ratio of A B to B C.

The three functions which we must use are—

$$\text{Sine A (usually written sin A)} = \frac{\text{B C}}{\text{A C}}$$

$$\text{Cosine A (usually written cos A)} = \frac{\text{A B}}{\text{A C}}$$

$$\text{Tangent A (usually written tan A)} = \frac{\text{B C}}{\text{A B}}$$

Measurement of Angles.—Angles are generally expressed by the number of degrees which they contain.

Let A B (fig. 2) represent a ruler pivoted at A and free to rotate about A in the plane of the paper. If it describe a complete circle it is said to have described an angle of three hundred and sixty degrees. If it move through a quarter of a circle it has described an angle of ninety degrees. Thus, the angle B A B″ is ninety degrees. Had it only moved through the ninetieth part of a quadrant it would have described an angle of one degree, which is the unit

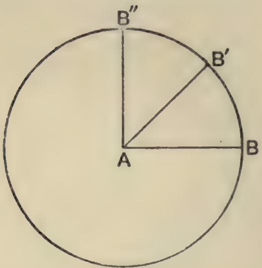


Fig. 2

angle employed in this system. Thus we see that the quadrant contains ninety degrees (generally written 90°). Further, each degree is subdivided into sixty minutes (written 60′), and each minute into sixty seconds (written 60″). This is what is called the sexagesimal method of measuring angles, and is in very frequent use. Another method of estimating angles is by their *circular measure*; the unit in this case is the angle whose arc is equal to the radius of the circle. To this we shall again refer when we come to speak of the enumeration of prisms.

Values of the Sine, Cosine, and Tangent.— These functions have definite values for any given angle. They have been computed and are to be found in any good set of mathematical tables. The method of finding the values in a few simple cases will be seen from the following examples. If in fig. 1 we actually measure the line B C in millimetres and the line A C in millimetres, and divide the number of millimetres contained in the former by that contained in the latter, we obtain a fraction which is the numerical value of the sine of the angle at A. In a similar manner we can obtain the numerical values for the cosine and the tangent of the angle.

Let A B C be an equilateral triangle, bisect the angle at B by the straight line B D then—

$$AD = DC$$

Since the $\angle$ ABD is half of the angle of an equilateral triangle, it contains 30°, and the angle at A contains 60°. Let us, for example, ascertain some of the functions of these two angles. To do this it is necessary to establish a relationship between the values of the sides AB, BD, AD. It has been shown that AD is equal to one-half of AC. As the triangle is equilateral, AD is equal to one-half of AB. Therefore AD and AB are to each other as 1 : 2. Moreover, by Euclid I. 47, we know that—

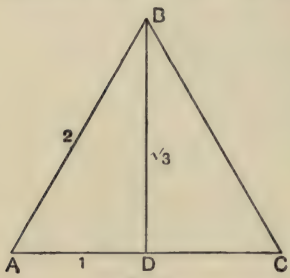


Fig. 3

$$BD^2 = AB^2 - AD^2,$$
$$\therefore BD^2 = 4 - 1, \text{ if } AD = 1.$$
$$= 3,$$
$$\therefore BD = \sqrt{3}.$$

Thus from the figure it is seen that—

$$\text{the tangent of } 30° \text{ is } \frac{1}{\sqrt{3}},$$
$$\text{the sine of } 60° \text{ is } \frac{\sqrt{3}}{2},$$
$$\text{and the cosine of } 60° \text{ is } \frac{1}{2}.$$

The student can easily with the aid of the figure supply the others.

Limiting Values of the Sine, Cosine, and Tangent.—In fig. 4 the sine of the small angle $B'OB$ is the length of the line $B'N$ divided by the length of the line $B'O$. If now we make the line $B'O$ rotate round the point O in the plane of the paper so as ultimately to coincide with OB, the line $B'N$ gradually diminishes, till when coincidence takes place its value is zero. Thus—

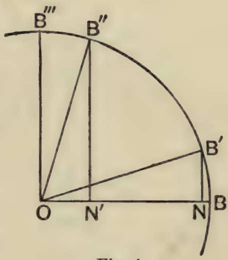


Fig. 4

$$\text{sine } 0^\circ \text{ is } \frac{0}{OB} = 0.$$

Again, if we consider the angle $B''OB$, we find that its sine is—

$$\frac{B''N'}{OB''}.$$

If now OB'' be made to rotate in the plane of the paper round the point O, the point B'' gradually approaches B''', and the point N' gradually approaches O. The nearer OB'' approaches OB''' the more nearly is $B''N'$ equal to $B''O$, and ultimately the difference between them becomes vanishingly small. Hence the value of the sine 90° is 1.

Similarly, the cosine of an angle of 0° is 1, and the cosine of an angle of 90° is 0. By analogous reasoning it can be shown that—

$$\text{tangent } 0^\circ = \text{zero,}$$
$$\text{tangent } 90^\circ = \text{infinity.}$$

Certain Properties of a Triangle.—Let ACB (fig. 5) be a triangle, and let the length of the side opposite the angle at A be denoted by a, that opposite the angle at B by b, and that opposite the angle at C by c. From C draw CD perpendicular to AB, then $\angle$CDA and $\angle$CDB are both right angles,

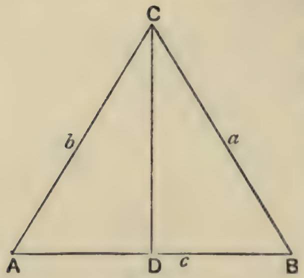


Fig. 5

$$\text{and } \sin A = \frac{CD}{b} \quad \therefore \; b \sin A = CD;$$

$$\text{also } \sin B = \frac{CD}{a} \quad \therefore \; a \sin B = CD;$$

$$\therefore \; a \sin B = b \sin A,$$

$$\text{and } \frac{\sin A}{a} = \frac{\sin B}{b}.$$

By drawing a perpendicular from A to BC, it can be shown that—

$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

$$\therefore \; \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Hitherto we have spoken of angles less than 90°. It is necessary to say something about the functions of angles greater than 90° but not exceeding 180°.

Trigonometrical demonstrations are beyond the purpose of this book, so we must be content by stating the following facts.

Let A B C (fig. 6) be any acute angle, then if C B be produced to D the angle ABD is called the supplement of A B C, and—

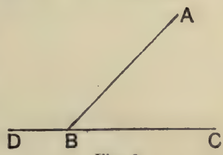


Fig. 6

$$\sin \text{A B D} = \sin \text{A B C},$$
$$\cos \text{A B D} = -\cos \text{A B C},$$
$$\tan \text{A B D} = -\tan \text{A B C}.$$

Calling the angle A B C a, the angle A B D = $(180° - a)$, and from what has just been said we have—

$$\sin (180° - a) = \sin a,$$
$$\cos (180° - a) = -\cos a,$$
$$\tan (180° - a) = -\tan a.$$

CHAPTER I

PROPERTIES OF LIGHT: REFLECTION

In this chapter an attempt will be made to discuss some of the more elementary properties of light, particularly those which are of frequent application in ophthalmology, and which ought therefore to be well understood by students of this branch of medicine. Briefly, it may be said that light is one of the modes by which the subject obtains knowledge of the objective world. It affects or sets in action the sense of sight, in virtue of which we may become cognizant of certain properties of bodies. Thus we ascertain the form and colour of an object with which we are not brought into actual contact.

The first thing to be noted is that light takes time to travel. This has been proved by several distinct methods of investigation, and the rate at which it travels has been measured with wonderful accuracy. Reference need only be made to the fact that Römer and Bradley each calculated the velocity of light from astronomical observations, and that it has also been determined by direct experiments made by Foucault, Fizeau, and others.

The results of these various investigations are in

closer agreement than might have been expected from the diversity of the methods employed. None of them differ much from 186,000 miles per second, which is therefore accepted as representing, with practical accuracy, the velocity of light.

Mode in which Light Moves.—Physicists recognize only two entities in the universe, viz. matter and energy, and the scientific world has only comparatively recently decided the question whether light is in its nature a transmission of matter or of energy.

If light consisted in the transmission of minute particles of matter thrown off from a luminous surface, the impact of these on the retina might be supposed to give rise to the sensation of light. On this assumption was based what is known as the **Corpuscular Theory of Light**, a theory which was held by Newton, and largely through his influence by the majority of scientific men long after his day. It is now, however, universally recognized that light is a form of energy, and is transmitted through space by means of wave motion. This form of transmission may be thus illustrated:—

If a stone be dropped into a pool of still water, waves are seen circling outward from the point of its immersion. Yet the water itself does not move bodily in the direction in which the waves are travelling. The initial disturbance consists of an up-and-down motion of particles of water. These communicate a similar motion to contiguous particles, and so on. Time is required for the com-

munication of the disturbance from particle to particle, and thus the vertical motions of the particles produce a succession of waves moving horizontally. The original motion or disturbance, therefore, spreads in the form of waves from the centre outwards. That the water has no horizontal motion may be proved by floating a cork at a little distance from the common centre of the circular waves. It will then be found that when a wave reaches the cork, the latter simply bobs up and down, but is not moved laterally along the surface of the water.

The energy of motion imparted to the water by the stone is thus transmitted from the point of immersion by means of waves of water. The water is in consequence a medium by means of which the energy is propagated. Other forms of energy also require media of some kind for their propagation. Thus sound can be transmitted through media, such as wood, iron, water, air, but cannot be propagated through a space void of matter, as is proved by the well-known bell-and-receiver experiment. Light, however, can pass through the receiver of an air-pump however perfect the vacuum, and, in fact, more readily than through air. Hence it is concluded that light does not require a material medium, in the common sense of the term, for its propagation, although it can pass through many material substances, those namely which we term transparent. Modern research has disproved the corpuscular theory, and established the undulatory theory of

light, *i.e.* that light is a form of energy which travels through space in undulations or waves. The essential medium of conveyance we have seen cannot be material in the usual signification of this term, and therefore scientists, on very solid grounds, hold it to be an imponderable elastic substance pervading all space, even the intermolecular spaces of bodies. To this medium the name *luminiferous* (*or light-bearing*) *ether* has been given—a term of too narrow significance, as late researches in electrical science have shown.[1]

Light being propagated by wave motion, it may be well to explain shortly the nature of waves and some of their characteristics.

Wave motion implies the ideas of wave length, of amplitude or wave height, and of wave speed or rate of propagation. When a stone is dropped into a pool, the eye can follow the course of the waves as they move over its surface, and in this

[1] The undulatory theory not only accounts in the completest manner for all the known phenomena of light, but by its means results have been predicted which experiment afterwards confirmed.

The corpuscular theory, on the other hand, is in direct contradiction to fact. Two objections to it may be mentioned. First, we can hardly conceive that particles of matter, however small, moving with the great velocity of light, could impinge on so delicate an organ as the eye without injury to it. Again, the fundamental assumption of the corpuscular theory leads very simply to the conclusion that the velocity of light is greater in denser than in rarer media—greater, for instance, in glass than in water, greater in water than in air—a conclusion which is found to be contrary to fact. Thus a particle of light (assuming the corpuscular theory) passing from a rarer to a denser medium (as from air to water) is, on approaching indefinitely near the common surface, attracted more strongly by the denser than by the rarer medium. The resultant attraction is perpendicular to the interface, and increases the velocity in that direction, the component parallel to the interface being unaltered. The resultant velocity of the particle is thus increased on entering the denser medium.

particular case it would not be very difficult to find the space traversed by any one wave in unit time. The waves of light, on the other hand, travel with enormous rapidity, viz., at the rate of 186,000 miles per second. The waves on the surface of the pool have a length which is quite appreciable. Those of light are extremely small, but have been measured by means which we shall afterwards indicate. The annexed diagram may be taken as roughly representing a wave.

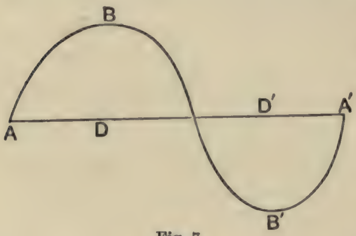


Fig. 7

The distance between any two points in the same phase[1] is defined as the wave length. Thus, in fig. 7, A A′, or twice the horizontal distance between B and B′, represents the wave length. The distance of the highest or of the lowest point of the wave from the line A A′ is called the amplitude of the wave. Investigations have shown that the colour of light depends upon the length of the wave, while its intensity depends on the amplitude. Thus, for one special kind of red light (that of the B line of the spectrum), the length in millimetres is 0·0006867, while for another kind of red light, that corresponding with the C line of the spectrum, it is 0·0006562.

[1] Two particles are said to be in the same phase when they are moving in parallel directions with the same velocity at the same instant.

Every wave motion is said to have a certain *frequency*. By this is meant that a certain number of waves pass a particular point in space per unit of time. Let us suppose two points in space to be separated from each other by a distance exactly equal to that which is traversed by a single wave pulse in unit of time. If now the disturbance start from the first point, it will reach the second point at the end of the first unit of time. If the exciting cause has continued to act during this period there will be a succession of waves of similar length extending between these two points. Further, let the number of waves between the two points be denoted by n, it is obvious that n waves have started from the first point in unit of time. This number n is called the frequency of the wave motion. If λ denote the wave length, the disturbance during the first unit of time has travelled a distance equal to $n\lambda$. Hence the velocity is equal to the wave length multiplied by the frequency, which is denoted by the equation—

$$v = n\lambda \quad \text{or} \quad n = \frac{v}{\lambda}.$$

As the velocity of light is known, and as the wave lengths for various colours have been measured, the corresponding frequencies can be calculated from the above formula.

Movement of Fluid Particles.—It has already been mentioned that the particles of water are not

displaced laterally to any appreciable extent, but simply undergo a movement of elevation and depression, *i.e.* the individual particles of water move at right angles to the direction of wave propagation. The same is true of the waves of light; the ether particles move at right angles to the direction in which the light is travelling, and the vibration is said to be transverse.

The simplest form of wave motion is when the particles of the medium move in straight lines with what is called a Simple Harmonic Motion.

Simple Harmonic Motion.—Let a point R move in the circumference of the circle A B C (fig. 8), in the direction shown by the arrow, and with uniform velocity. Let R be the position of the point at any instant, and draw R S perpendicular to a fixed diameter A B. As R moves uniformly in the circumference, the foot of the perpendicular describes a simple harmonic motion in A B. The relative distances passed over in equal times by M can be graphically represented by dividing the circumference into equal arcs A R, R R_1, R_1 R_2, . . . , and drawing the perpendiculars R S, R_1 S_1, R_2 S_2 . . .

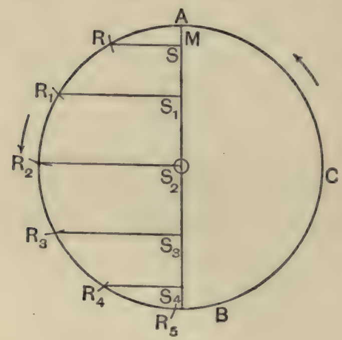


Fig. 8

Thus R moving uniformly describes the equal arcs AR, RR_1, R_1R_2, . . . in equal intervals of time; while M, by condition, simultaneously describes the unequal lengths AS, SS_1, S_1S_2, . . . It will be clear from the diagram that M starts from rest at A, increases its velocity gradually until it arrives at O, the centre of the circle, where its velocity is momentarily equal to that of R. From O onwards to B its velocity gradually decreases; and starting now from B, the path BA is traced in like manner. The point M is then said to execute a simple harmonic motion.

An excellent illustration of what is meant by a simple harmonic motion is obtained from the study of a pendulum. When the bob of the pendulum is drawn a little to one side and then released, gravity at once attracts the bob towards the earth, the rod of the pendulum prevents its falling, and compels it on account of its constant length to describe an arc of a circle, which if the motion be very small is practically a straight line. The velocity, however, is not uniform, but gradually increases till the bob is at its lowest point, and then gradually decreases till it becomes zero on the other side.

Wave Front and Ray.—If an extremely small plane surface be introduced at any part of a wave perpendicularly to the direction in which the wave is travelling, it will approximately occupy a wave front. All the particles in a wave front are moving

in the same direction and with the same velocity. It should be observed, however, as can easily be seen in the case of waves in a pool, that if a portion of a wave be considered when it is very near the centre of disturbance, even a small portion of its front is curved and not plane. It is only when the radius of curvature is very large that a small portion of the wave front approximates to a plane.

By the term *ray* is meant the direction in which a particular point of the wave front is travelling. It is at right angles to the wave front.

Interference.—If two waves meet each other in the same phase—for example, if two crests or two hollows meet—then the disturbing effect is double of what it would be if only one were present. If, on the other hand, two waves meet, so that the hollow of one corresponds with the crest of the other, the two neutralize each other, and there is no apparent disturbance. Waves producing such effects as these are said to interfere. The rectilineal propagation of light depends on interference. This property is also of importance in measuring wave lengths, which is conveniently done either by Fresnel's bi-prism or by a diffraction grating.

When the ether waves which constitute light impinge upon any surface then certain changes take place. Let us suppose, in the first place, that they impinge upon an *opaque* surface. The changes which may then occur are that the light is scattered, absorbed, regularly reflected, or polarized. At

every opaque surface all these changes take place, but some more prominently than others. If, for example, the surface be black, almost all the light is absorbed, and consequently little of it is reflected. On the other hand, if the surface be red, almost all the components of the light are absorbed except the red elements, and these are reflected. On examining a red surface seen by white light with the spectroscope it is found that some of the other components of the white light are reflected to a slight degree, but that the red components predominate. Those which are not reflected, or only imperfectly so, are wholly or partially absorbed by the substance. It is not our intention to deal with polarization at all, beyond saying that polarized light differs from ordinary light in having all its vibrations in one plane, and that all reflected light is more or less polarized. In the second place, if the ether waves impinge upon a *transparent* surface, they are in part reflected and in part transmitted through the substance. Thus, if an observer looks at a sheet of glass he can see his own image reflected by the glass, while at the same time another observer can view him through it. Under certain circumstances transmitted light may also be polarized.

Reflection of Light.—When light is reflected from a surface it is turned back into the same medium. There is no change in velocity, but there is a change in direction.

Let A B (fig. 9) be a reflecting surface, and let C O indicate a narrow beam (pencil) of light incident at the point O, and let N O be perpendicular to the surface A B. N O is called the normal to the surface at O, C O is called the incident pencil or ray, and the angle C O N is defined as the angle of incidence. By experiment it has been proved that the light is reflected in the direction O D, so that the angle C O N is equal to the angle D O N. The angle D O N is called the angle of reflection. The laws of reflection are as follows: (1) The incident and reflected rays are in the same plane as the normal at the point of incidence; (2) the angle of incidence is equal to the angle of reflection.

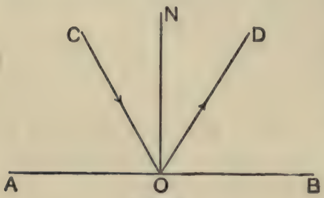


Fig. 9

Mirrors, as usually employed, are made of silvered glass or of speculum metal. Such mirrors have the property of reflecting all the components of the incident light; consequently, if the incident light be white, the reflected light is also white.

The Plane Mirror.—A plane mirror is one whose surface is flat.

Let P M (fig. 10) represent such a mirror in section, and let O be a point of an object in front of this mirror in the plane of section. Draw O P perpendicular to P M, and produce it to x. Let O B be a

ray of light incident on the surface of the mirror at the point B; then NBO is the angle of incidence, and NBD is the angle of reflection, BN being the normal or perpendicular at B. After reflection the light will take the direction of BD. Produce BD backwards

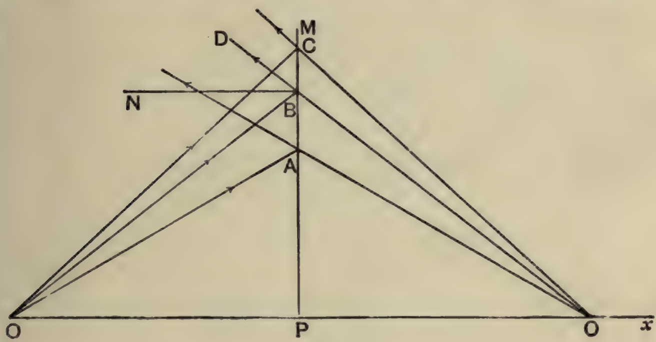


Fig. 10

to meet OP in O′. The light after reflection will appear as if it came from O′, which is as far behind the mirror as O is in front of it. To prove this, consider the two triangles BPO and BPO′. The angles BPO and BPO′ are right angles (by construction),

$$\therefore \angle BPO = \angle BPO',$$

$$\text{and } \angle NBO = \angle BOP,$$

$$\text{and } \angle DBN = \angle BO'P;$$

$$\text{but } \angle NBO = \angle DBN,$$

being the angles of incidence and reflection respectively,

$$\therefore \angle BOP = \angle BO'P.$$

$\therefore$ in the two triangles we have

$$\angle BOP = \angle BO'P,$$
$$\text{and } \angle BPO = \angle BPO',$$
$$\text{and the side } BP \text{ common,}$$
$$\therefore OP = O'P$$

Hence O' is at the same distance behind the mirror that O is in front of it. Similarly, it can be shown that all rays coming from O appear after reflection to come from O', which, therefore, is virtually the source of light, so far as the reflected rays are concerned, and O' is called the image of O.

Images.—Images are of two kinds—(1) real, (2) virtual. Rays of light from a luminous point may, by reflection, or refraction, have their directions altered so as—

(1) To intersect. The point of intersection is a real image of the luminous point, and may obviously be received on a screen.

(2) Not to intersect, but to appear to proceed from a different point. This apparent source is a *virtual* image of the luminous point, and is the intersection of the rays produced backwards. Such an image cannot be received on a screen, although under suitable circumstances it can be perceived by the eye.

A point source in front of a plane mirror has, as shown above, its image behind the mirror, and inspection of the diagram proves that the image is virtual.

The figure formed by the images of all the points of an object is the image of that object, and is real or virtual according as the images of the points are real or virtual.

Real images, whether produced by reflection or refraction, are always inverted. Virtual images are always erect. Thus, in the case of a plane mirror, if a person look at himself in an ordinary mirror he will see an erect, not an inverted, image. At the same time, however, it is a laterally inverted one, for if the observer moves his right hand the image will appear to move its left.

The Concave Mirror.—As concave mirrors are very often employed in the construction of ophthalmoscopes and other instruments used in medicine, some detailed account must be given of their optical properties. They generally consist of segments of

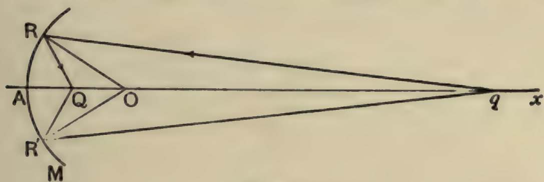


Fig. 11

hollow glass spheres silvered on the outside, so that the hollow side reflects.

Let M A R (fig. 11) be a section of such a mirror, O the centre of the sphere, called also the centre of curvature of the mirror; the line A O produced both

ways is called the principal axis of the mirror. Take any point q on the axis A O, and from q as a source let a very thin pencil of light be incident on the surface of the mirror at the point R. Join R O. If the pencil of light be so thin that the part of the surface of the mirror on which it is incident is very minute, that part may be regarded as a small plane surface, and R O being a radius of the sphere is normal to the surface at R. Hence the law of reflection at a plane surface enables us to trace the direction of a reflected ray. The angle of incidence q R O is equal to the angle of reflection Q R O.

Since, therefore, $\angle \mathrm{QRO} = \angle q\mathrm{RO}$, we have

$$\mathrm{QO} : q\mathrm{O} :: \mathrm{QR} : q\mathrm{R} \text{..........Euclid, VI. 3.}$$

Now, if the arc A R is small compared with A Q, the distance q R is nearly equal to q A, and the nearer R approaches to A the more exact is this approximation. Therefore without sensible error we may write the above proportion—

$$\mathrm{QO} : q\mathrm{O} :: \mathrm{QA} : q\mathrm{A},$$

$$\text{i.e. } \mathrm{AO} - \mathrm{AQ} : \mathrm{A}q - \mathrm{AO} :: \mathrm{AQ} : \mathrm{A}q.$$

Calling A q, u; A Q, v; and A O, r, we have—

$$r - v : u - r = v : u,$$

$$\therefore ur + vr = 2vu.$$

On dividing this equation by uvr we obtain—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

If now a small pencil of light be incident on the mirror in the direction q R′, so that the arc A R′ is also very small compared with A Q, it is obvious that after reflection this ray will also pass through the point Q.

With the same limitation it can be shown that all rays coming from q, incident on the mirror, pass approximately through the point Q. Therefore at Q a real image of the point of the object at q is formed. As the image is real, it is also inverted.

Pairs of points on the axis, such as Q and q, whose distances from the mirror satisfy the relation

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r},$$

are said to be foci conjugate to each other. If one is the position of an object, the other is the position of its image. Thus, if the object be situated at q an inverted image of it, which could be received on a suitably arranged screen, is formed at Q, while, on the other hand, if the object be placed at Q, an inverted image of it is formed at q.

If now the source of light at q be taken to a great distance the angular divergence of the light may be disregarded, for the angle R q A is then extremely small, and q R is practically parallel to q A. When this is the case, Q, which takes up a definite limiting position, is called the *principal focus*, and A Q the *principal focal length* of the mirror. It is also sometimes called the *focal length*

of the mirror. Returning for a moment to our last formula—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r},$$

we may regard u as being infinite compared with v. Indeed, if the rays q R and q A are absolutely parallel to each other, then the distance of q must be infinite. Any finite quantity divided by an infinite quantity is zero, hence the equation becomes—

$$\frac{1}{v} = \frac{2}{r}.$$

This, as already seen, is a special case, and the focus which is formed by a pencil of parallel light incident on a mirror in a direction parallel with the axis is called the principal focus. The principal focal distance, or focal length, is generally indicated by F, and we have—

$$\frac{1}{\text{F}} = \frac{2}{r},$$

which means that the focal length is half the radius of curvature.

Practical Measurement.—To find the focal length of a concave mirror select as an object something distant, such as a lighted candle at least twenty feet away, or, better still, the chimneys of a house at a considerable distance. With the mirror throw an image of the selected object on a suitable screen, such as a sheet of writing-paper, and measure

the distance between the screen and the mirror when the image is quite sharp. The distance so measured is approximately the focal length.

The same thing can be obtained directly by means of the spherometer, an instrument with which we are able to measure the radius of curvature of spherical surfaces.

Figure and Description of a Spherometer.— The spherometer consists of a disc of metal with a graduated circumference, and supported on three legs made of hard steel rounded at the points. The legs are of equal length, and are equidistant from each other. The centre of the disc is pierced by a steel screw, the point of which is also rounded. This screw serves as a fourth foot. When the instrument is set on a plane glass plate resting on the three fixed legs, no rocking takes place when slight movements are imparted to the instrument with the hand. Should, however, the fourth leg be made slightly longer than the other three then rocking at once takes place. A little practice soon enables an experimenter to determine with great exactness when the fourth point is just touching the surface. The

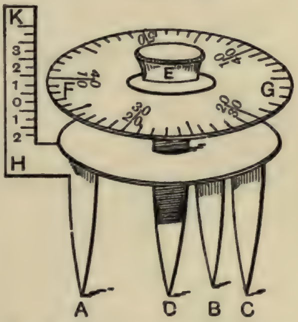


Fig. 12 (From Glazebrook and Shaw)

screw of a good spherometer has a thread with, say, 100 turns to the inch. One complete turn therefore means that the fourth leg has either been raised or lowered by $\frac{1}{100}$ of an inch. By means of the scale on the disc of the instrument any fractional part can be measured with great accuracy. Thus, suppose that the circle is turned through an angle of 60°, equal to $\frac{1}{6}$ of the entire circumference, then the fourth leg is, according to the direction of the movement, raised or lowered by $\frac{1}{600}$ of an inch. Thus very fine measurements can be made.

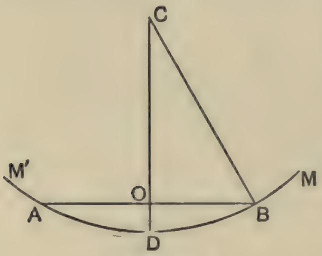


Fig. 13

The instrument is used in the following manner for measuring radii of spherical surfaces:—

Let M D M′ (fig. 13) be a concave surface, and let us suppose that after applying the spherometer to a plane surface, and carefully adjusting it to that plane surface, so as to get the zero reading, we transfer the instrument to this concave one, so that one foot rests on A and another on B. The middle screw is now turned till it just touches the concave surface at the point D. The length O D is measured by the number of turns and fractions of a turn which have been given to the screw in order to bring its point from O to D. Moreover, the distance

OB is also known, for it is the distance between the point of the screw when applied to a plane surface and any one of its three feet. Let OD = h, and OB = m, and CB and CD each = r.

Then in the triangle COB—

$$r^2 = m^2 + (r - h)^2,$$
$$r = \frac{m^2 + h^2}{2h}.$$

As both m and h are known, r, the radius of curvature, is thus found.

In measuring the radius of curvature of a convex surface, the spherometer is, as in the previous example, first applied to a plane surface. It is then transferred to the convex surface. The point is now gradually raised by means of the screw till the three feet and the point just touch the surface. The distance through which the screw has been turned gives us, as before, the value of h, and m is a constant for the instrument used.

The Geneva lens measurer is an instrument much used by spectacle vendors, and sometimes even by ophthalmic surgeons, to determine the focal length of thin biconvex and concave lenses. It is based on the principle of the spherometer, and consists essentially of two legs of equal and constant length parallel to each other. Exactly half-way between them is a third leg, which, acted upon by a spring, is capable of movement in a direction parallel to the others; all three are in the same plane. The

movable leg is connected with an indicator which turns round a dial. To determine the radius of any curved surface the three points are applied firmly to the surface, when the middle leg is pushed in to an extent depending upon the amount of convexity or concavity. When this is done the indicator on the dial will be found to have moved to a new position.

It is taken for granted that a certain curvature corresponds to a given focal length, and consequently the dial is marked to indicate the refractive power of the lens which is being tested. This assumption, however, is not even approximately true.

Later it will be shown that if F be the focal length of a thin convex lens, then—

$$\frac{1}{F} = (\mu - 1)\left(\frac{1}{R} + \frac{1}{R'}\right),$$

where μ is the index of refraction of the glass of which the lens is made, and R and R′ are the numerical values of the radii of curvature of its surfaces. The Geneva lens measurer gives very accurately the values of R and R′, but does not determine that of μ. It is a valuable instrument for determining the curvatures of lens surfaces, but is not to be relied upon for information as to their focal lengths.

Images formed by Concave Mirrors. — In fig. 14, P X R is a concave mirror; X X′, passing

through the centre of curvature C and the principal focus F, is the principal axis, and A B a luminous object in front of the mirror. To find the image of the point B, the following simple construction suffices. Since from the point B light is sent out in all possible directions, one ray must pass through the point C, and will be incident on the mirror at the point R. As it is incident in the direction of one of the radii of curvature, it strikes the surface

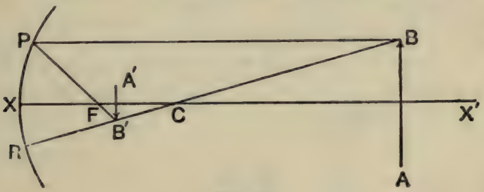


Fig. 14

normally, *i.e.* at right angles. It is therefore reflected back along the same line as it came. The image of B is thus formed somewhere on the line BR. To obtain its exact position it suffices to find the point at which another ray coming from B intersects the first one after reflection. Let the ray B P parallel to the axis be taken. It has been shown at page 23 that after reflection this ray passes through the principal focus F, and will therefore intersect the first ray at some point B′, which is consequently the image of B.

In the same manner it can be shown that A′ is the image of A, and similarly that all points of the object A B have corresponding images in A′B′. Thus A′ B′ is a real and an inverted image of A B. It is

apparent from the diagram that the image is smaller than the object. Were the object placed at A′ B′ a magnified image of it would be formed at A B.

If a self-luminous object, such as a candle flame, be placed at the focus of the mirror, the light is reflected in parallel pencils (see page 23). Should it, however, be nearer to the mirror than the principal focus, then the rays of light after reflection continue to diverge, and do not form a real image, but appear as if they proceeded from a virtual image behind the mirror.

In the diagram, fig. 15, let P M be a concave mirror whose focal length is X F, and let the object A B be

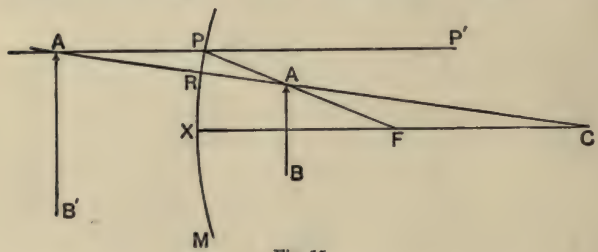


Fig. 15

placed between the mirror and its principal focus. A construction similar to the last enables us to get the image of any point of this object.

Consider the point A from which rays of light proceed in all possible directions. One ray A P must proceed as if it came from the principal focus. After reflection it leaves the mirror in the direction P P′ parallel to the axis. Another ray

A R must leave the point A in the direction of the radius C R. After reflection it is sent back in the direction R C. These two rays therefore do not meet, but after reflection from the mirror diverge from each other. Were these reflected rays prolonged backwards through the mirror they would meet at the point A′, which is therefore the virtual image of A. To an observer placed in front of the mirror the reflected image of the point A would appear to be situated at the point A′. In the same way it can be shown that B′, a point in A′B′, is the virtual image of B, and A′B′ of A B. Thus, when an object is placed in front of a concave mirror at a less distance than its principal focal length, the reflected image is virtual, is erect, and is magnified.

Convex Mirror.—A convex spherical mirror is one in which reflection takes place at the convex surface.

Let P A M (fig. 16) represent a convex mirror in section, the axis of which is O Q. Let a ray of light from Q be incident at the point P, and let O be the centre of curvature of the mirror. The line O R, being a radius, is at right angles to the surface of the mirror at the point P. The ray after reflection travels in the direction P S, making the angle of reflection R P S equal to the angle of incidence Q P R. It is thus obvious that the reflected ray does not again cut the axis of the mirror, but that if this ray be prolonged backwards through the mirror it will cut the axis at the point q, which is therefore

the image of Q. It will be observed that the distance A Q is measured *from* the mirror in a direction

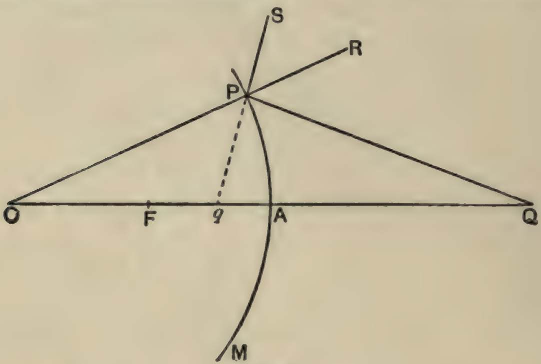


Fig. 16

opposite to those of A q and A O. It is therefore necessary to regard these two latter distances as negative, if we consider A Q positive. Thus we get—

$$\text{AQ} = u, \ \text{AO} = -r, \ \text{A}q = -v.$$

And as before,

$$q\text{O} : \text{QO} :: q\text{P} : \text{QP};$$

$$\textit{i.e.}\ \text{AO} - \text{A}q : \text{AQ} + \text{AO} :: q\text{P} : \text{QP}.$$

Now, if the point P be brought very near the point A, qP is very nearly equal to qA, and QP to QA; and as v and r are both negative, we have, on substituting,

$$v - r : u - r :: -v : u;$$

$$\therefore (v-r)u = -v(u-r),$$

$$\therefore vr + ur = 2uv,$$

$$\therefore \text{dividing along by } uvr, \ \frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

It will be observed that the formula for the convex mirror is identical with that for the concave, and it is only necessary in working with these formulæ to remember that in the case of the convex mirror r and v are always negative quantities.

In speaking of concave and convex spherical mirrors it has only been necessary to discuss a particular case, namely, that in which the point of incidence is so close to the point at which the axis of the mirror intersects its surface that Q A differs in length from Q P (fig. 16) by a quantity which can be disregarded in practical work. The formula—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}$$

therefore, is only true for those rays which are incident in the immediate vicinity of the point A. Those which are incident at other portions of the surface are not reflected to q, but cut the axis of the mirror nearer to its surface. Strictly speaking, if a pencil of light, all the rays of which are parallel to the axis, is incident on the surface of a spherical mirror, only those near the axis pass through the principal focus; the others go to form a continuous series of foci extending from the principal focus towards the mirror. This departure from the simple law, found for a limited portion of the surface in the neighbourhood of A, is due to the form of the reflecting surface, and hence is called *spherical* aberration. It is thus obvious that the amount of light

at the focus is not so great as it would be were the entire parallel pencil reflected through it. To gain the maximum of illumination, parabolic mirrors are sometimes used in ophthalmology, especially in such operations as the inspection of the cornea.

Parabolic Mirror.—Let P M (fig. 17) be a part of a parabola. If the curve be rotated round its axis xx', P O will trace out a surface which, if polished on its inner side, will form a parabolic reflector. It can easily be shown that all rays of light parallel to the axis $x\,x'$, and incident on the surface of such a mirror, are reflected accurately through the point S, which is termed the focus of the mirror.

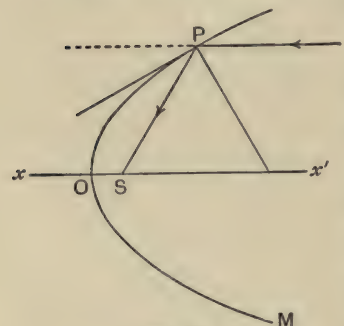


Fig. 17

Hitherto we have considered the reflection of pencils which are called direct, *i.e.* pencils whose geometric axes coincide with the geometric axes of the mirrors. The subject of oblique reflection, *i.e.* of pencils whose geometric axes do not correspond with those of the mirrors, is not one of any great importance to the student of ophthalmology. It is therefore omitted from the present work.

CHAPTER II

REFRACTION

Refraction of Light.—It is matter of common observation that a straight stick, partially immersed in water, in a direction oblique to the surface, appears to be bent at the point at which it enters the water; moreover, the part immersed seems to

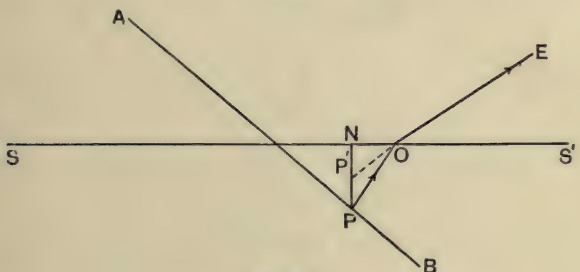


Fig. 18

be bent *towards* the surface of the water. The explanation of this phenomenon is at once apparent from a study of fig. 18, in which A B represents the stick, and S S′ the surface of the water. Let P be a point on the stick capable of reflecting rays of light in all possible directions. P O is one of these rays, and as it leaves the water it is bent out of its original course, so that an eye placed at

E sees this ray as if coming from P′. The point P′ is nearer to the surface than P, and P appears to the eye as if it lay in the direction E O P′. This abrupt change in the direction of a ray on crossing the bounding surface of two different media is termed refraction.

Another simple method of illustrating the same thing is to place a sixpence or any small object

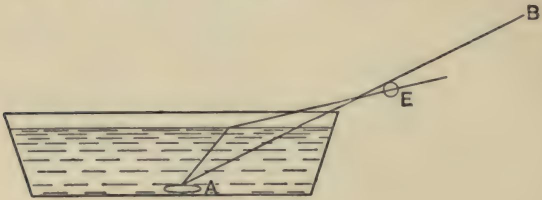


Fig. 19

in a cup or other suitable vessel; the observer then places his head so as just to lose sight of the sixpence behind the edge of the cup. If water be now poured into the vessel the sixpence again reappears though the observer has not changed his position. Reference to fig. 19 sufficiently explains this phenomenon. A B is a ray of light coming from the sixpence to the edge of the vessel, and passing on, but at such an angle that it is not received by the observer's eye at E. After some water has been poured in, another ray, A O, is refracted at the surface of the water, and passes the edge of the dish in a direction which enables the observer to see the coin.

Let S (fig. 20) be the interface of two refracting media such as air and glass, and let N P be the normal at P to S. A pencil of light incident at P in the direction LP does not, on entering the glass, continue in the same straight line, but is bent towards the normal P N′ in some such direction as P L′. The angle L P N is called the angle of incidence, and the angle L′ P N′ is called the angle of refraction. They are in the same plane, and as we shall presently see, the sine of the angle of incidence always bears a constant relationship to the sine of the angle of refraction for any two media.

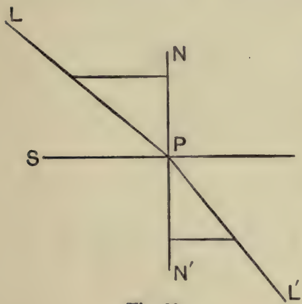


Fig. 20

Refraction depends on Change in Velocity.—When light passes from one transparent medium into another of greater density, the velocity of propagation is diminished. On this depends the bending of the ray when the incidence is oblique. If, however, the incident pencil be at right angles to the surface separating the two media, no bending takes place, but the change of velocity is the same, being, in fact, independent of the angle of incidence. Some of the energy is reflected back into the first medium, and another portion of it is transmitted into the second. This is true for every angle of incidence.

If the light propagated in one transparent medium is incident on the surface of another of greater density, some of the light is reflected back into the first medium, making an angle of reflection equal to the angle of incidence, and another portion passes into the second transparent medium. If the angle of incidence is greater than zero, the second portion not only enters the second medium, but its direction is also changed.

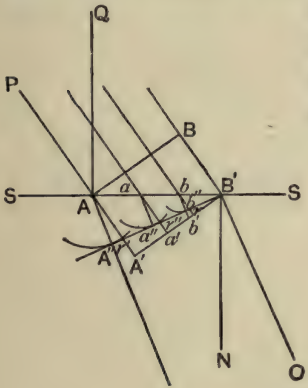
Before proceeding to a proof of the fact that the change in direction can be accounted for by a difference in the rates at which light travels in the two media, it will be necessary to refer again to the idea of a *wave front*. The simplest case is that of a pencil of parallel rays of light. If a plane be taken at any point at right angles to the direction of propagation, that plane is a wave front.[1]

Fig. 21 shows how the change in the direction of the wave front depends upon the change in velocity.

Here we have a pencil of parallel light incident on the surface S S of the second medium, supposed of greater density than the first. Of this pencil A B is a wave front. Let us suppose that the velocity with which light travels in the first medium is v, and in the second v', and let it also be assumed that $v : v' :: 3 : 2$ (which is very nearly true if the first

[1] At the wave front all the particles of the luminiferous ether are in the *same phase.*

medium be air and the second glass), then the following simple construction will suffice. Let A A′ be the distance which the light incident at A would have gone had there been no second medium during the interval taken by the disturbance to travel from B to B′; then A A′ = B B′. The portion of light incident at A has, however, gone a less distance than



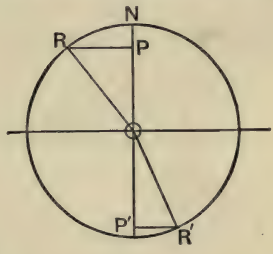


Fig. 21

A A′, on account of the greater density of the second medium. Since $v : v' :: 3 : 2$, the disturbance in the second medium has only gone $\frac{2}{3}$ of A A′, while that in the first medium has been travelling from B to B′. Now, when the light was first incident at A, it set up a disturbance in the second medium, which extended from A as centre in a spherical wave. Let us take A r' = $\frac{2}{3}$ A A′, and with radius A r' and centre A describe a sphere represented in section by a circle. This marks the limit of the disturbance in the second medium at the instant that the disturbance in the first medium has arrived at B′. In

like manner, if a radius $a\,r''$ is taken $= \frac{2}{3}$ of $a\,a'$, another sphere represented in section by a circle can be drawn marking the limit of the disturbance caused by the portion of the original wave front incident at a at the same instant as before. This portion, had there been no second medium, would have reached a'. The common tangent plane of all these spheres represented in section by A″ B′, is the new wave front, and the parallel lines A A″, B′O, $a\,a''$, $b\,b''$, drawn at right angles to the wave front, indicate the direction which the light takes after refraction.

From B′ draw B′N at right angles to S S; the $\angle$ N B′O is defined as the angle of refraction. It is obviously equal to the $\angle$ A″B′ A. Again, draw A Q at right angles to S S, then $\angle$ P A Q is the angle of incidence, and it is $= \angle$ B A B′.

From the figure therefore we see that the sine of the angle of incidence (i) is—

$$\frac{\mathrm{B\,B'}}{\mathrm{B'\,A}},$$

and the sine of the angle of refraction (r) is—

$$\frac{\mathrm{A''\,A}}{\mathrm{A\,B'}},$$

$$\frac{\sin i}{\sin r} = \frac{\mathrm{B'\,B}}{\mathrm{A\,A''}}.$$

Now B B′ is proportional to v, the distance travelled in air in unit of time, and A A″ is pro-

portional to v', the distance travelled in the second medium in the same time,

$$\text{therefore } \frac{\sin i}{\sin r} = \frac{v}{v'}.$$

Since the velocity of light in air may be regarded as a constant, and also the velocity in any other medium, the ratio $\frac{v}{v'}$ is constant for these media. This relation, which is generally expressed by the Greek letter μ, is called the index of refraction, relatively to air, of the substance composing the medium. The subjoined table gives the indices of refraction of a few transparent substances, that of air being unity:—

Diamond	2·43
Glass (two lead and one flint)	1·8
Flint Glass	1·57–1·6
Crown Glass	1·52
Water	1·336
Aqueous Humour	1·337
Vitreous Humour	1·339
Crystalline Lens	1·337
Canada Balsam	1·54

The law expressed in the formula—

$$\frac{\sin i}{\sin r} = \mu \text{ (a constant)},$$

frequently written, $\sin i = \mu \sin r$, is from its discoverer termed Snell's Law, and holds for light

passing from one medium into a second of different density for every value of the angle of incidence.

The Critical Angle and Total Reflection.—Let B C (fig. 22) be the surface of a sheet of water

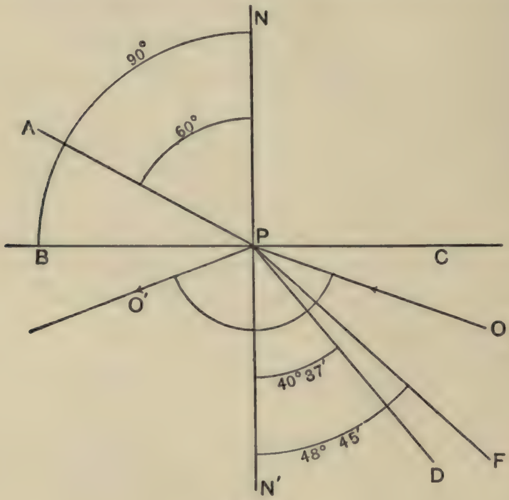


Fig. 22

of index of refraction 1·33, and let the superimposed medium be air, of which the index of refraction may, for our present purpose, be taken as unity.

Let A P be a ray of light incident at the point P, making with the normal, P N, an angle of incidence, A P N, equal to, say, 60°. The corresponding angle of refraction can be calculated by the above for-

mula, and is found to be about 40° 37′. The ultimate direction of A P is thus P D.

Now, if the angle of incidence be very nearly a right angle, which is the case when the light just skims along the surface of the water, the corresponding angle of refraction, as calculated by the above formula, is 48° 45′, and the ultimate direction of such a ray of light entering the water at the point P is P F. Further, if light were leaving the denser medium in the direction F P it would, on emergence into the rarer medium, just skim along the surface of the water. The angle F P N′ is termed the *critical angle* for water.

Again, light may be propagated in the second medium in such a direction as O P; in this case it would not pass out of the water at all, but would be totally reflected in the direction P O′, and the angle of incidence O P N′ would be equal to the angle of reflection N′ P O′. In fact, in the second medium light incident on the surface of separation at any angle greater than 48° 45′ would be totally reflected, and in every case the angle of incidence and the angle of reflection would be equal. The critical angle can easily be obtained for any transparent substance from the formula—

$$\sin i = \mu \sin r,$$

$$\frac{\sin i}{\mu} = \sin r.$$

To obtain the critical angle for any medium, the

angle i is to be taken as equal to 90°, and the sine of 90° is equal to 1;

$$\therefore \frac{1}{\mu} = \sin R,$$

where R indicates the critical angle. The critical angle for glass of index of refraction 1·5 is 41° 49′.

The phenomenon of total reflection is taken advantage of in the rectangular prism, an instrument of importance in the construction of several pieces of ophthalmic apparatus. In fig. 23, A B C represents a rectangular isosceles prism in section. The angle at B is a right angle, and the angles at A and C are each 45°. D F represents a pencil of parallel light incident at right angles to the surface at A B. As the angle of incidence is 0°, this pencil is not bent at the surface A B, but passes into the prism in the same straight line, and is incident on the surface A C at the point P. Let P N be drawn perpendicular to the surface A C, then the angle of incidence at this surface D P N is equal to 45°. This is greater than the critical angle for glass, and therefore the light is totally reflected at the point P in the direction P H, and thus leaves the prism at right angles to its original direction.

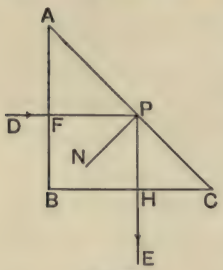


Fig. 23

Absolute and Relative Indices of Refraction.—Were light, propagated in a vacuum, incident

at an angle i on the surface of a transparent substance, on entering the substance it would be refracted at an angle r, such that $\frac{\sin i}{\sin r}$ would be equal to μ, which is in this case termed the absolute index of refraction for the substance under consideration. The index of refraction of vacuum is, strictly speaking, unity but as it only differs from that of air by ·000294, there is no sensible error in taking the index of refraction of air as 1.

If a ray of light impinge in an oblique direction on the interface of any two transparent media, having different indices of refraction, the ray will be refracted in passing from one to the other. If v_1, v_2 be the velocities of light in the first and second media respectively, then the construction on page 39 shows the abrupt change in the direction of the ray on entering the second medium.

Let us suppose that the first substance is glass and the second water, then the index of refraction of glass relatively to water, or of water relatively to glass, is easily obtained from the following considerations:—

Let the absolute index of refraction of glass be $\mu_1 = \frac{v}{v_1}$ where v is the velocity in vacuum, and v_1 the velocity in glass. Similarly, $\mu_2 = \frac{v}{v_2}$ may represent the absolute index of refraction of water, v_2 being the velocity in water. Hence we have—

$$\mu_1 = \frac{v}{v_1} \text{ and } \mu_2 = \frac{v}{v_2}.$$

On dividing the first equation by the second, we have $\frac{v_2}{v_1} = \frac{\mu_1}{\mu_2}$, which expresses the relative index of refraction for water to glass.

Let us suppose that we have three transparent media, A, B, and C (fig. 24), of different indices of refraction, with parallel surfaces superimposed upon

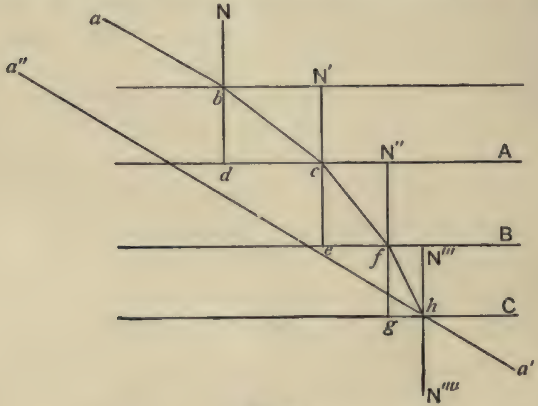


Fig. 24

each other, and the index of refraction of C greater than that of B, and that of B greater than that of A. The figure gives a diagrammatic representation of the passage of a pencil of light through the plates. At the first surface the angle of incidence abN has the corresponding angle of refraction dbc. This angle of refraction is equal to the angle bcN', for bd is parallel to $N'c$.

The angle bcN' is the angle of incidence at the

second surface, and the corresponding angle of refraction is *ecf*, which in turn is equal to the angle of incidence at the second surface, and so on for any number of substances. The final angle of refraction, *i.e.* the angle of emergence into the original medium, can be shown to be equal to the angle of incidence at the first surface. Hence, a pencil of light, after passing through several plates with parallel surfaces, is, after the last refraction, parallel to its original direction, that is, there is no deviation. There is, however, displacement of the transmitted ray, for in the diagram the ray $h\,a'$ appears as if coming from a point a'' and not from a.

In the experiment originally quoted, of a rod thrust into water (see page 35), we have seen that the rod appears to be nearer to the surface than it is in reality. The true depth divided by the apparent depth measures the index of refraction.

Measurement of Index of Refraction.—There are several methods of measuring indices of refraction, but three of them are of special importance.

Fig. 25

These are measurements (a) by the microscope, (b) by the angle of total reflection, (c) by means of prisms.

(a) *By the Microscope.*—In the annexed figure (fig. 25), M is the object-glass of a microscope, S is a

slide, and C is a cover-glass which rests upon two pieces of platinum wire, forming a kind of cell. On the upper surface of the slide, and on the lower surface of the cover-glass slight scratches are made by means of a diamond. The observer then places the cover-glass *in situ*, making the scratch on its surface as nearly as possible vertically above the scratch on the slide. To measure the distance between the slide and the cover-glass, he first accurately focuses the scratch on the slide. He then carefully turns the fine adjustment till the scratch on the under surface of the cover-glass is in focus. By means of the scale which is engraved on the fine adjustment of every good microscope he can measure accurately the number of turns and fractional parts of a turn which have been necessary to raise the object-glass of the microscope through the same distance as the depth of the cell. In this way the exact depth of the cell is ascertained. The next step is to remove the cover-glass, and to place on its under surface a few drops of the fluid whose index of refraction is to be measured. When this is done the cover-glass is again placed in position and the vertical distance between the two scratches is again measured in the same manner. This will now appear to be less than in the former experiment, for the same reason that the part of a stick immersed in a liquid appears to be nearer to the surface than it is in reality. Thus we get the measurement of the apparent depth. The true

depth divided by the apparent depth is the index of refraction for the liquid. It is convenient always to use the same pieces of wire and the same cover-glass and slide, for then the depth of the cell is a constant quantity.

(*b*) *By the Angle of Total Reflection.*—This method is also available when only a small quantity of the fluid can be obtained. It, however, requires special apparatus, and as it cannot be readily used without considerable trouble, it is not thought necessary to enter into a detailed description of this method here.

(*c*) *By means of Prisms.*—Before describing in detail this method it is necessary that the student should have some knowledge of the geometry of the prism. As prisms are of great utility in ophthalmology, both as aids to diagnosis and as important therapeutic agents in certain cases, all ophthalmic students should be acquainted with their properties.

Let A B C (fig. 26) be a section of a prism, and let P O be a small pencil of light incident at the point O on the side A B. Here it is refracted in some direction such as O O′, so that—

$$\sin \mathrm{PON} = \mu \sin \mathrm{XOO'}.$$

At the second surface on emergence it is again refracted, so that—

$$\mu \sin \mathrm{OO'X} = \sin \mathrm{P'O'N'}.$$

The original direction of the light was PY, and its direction after leaving the prism is YP′. It has

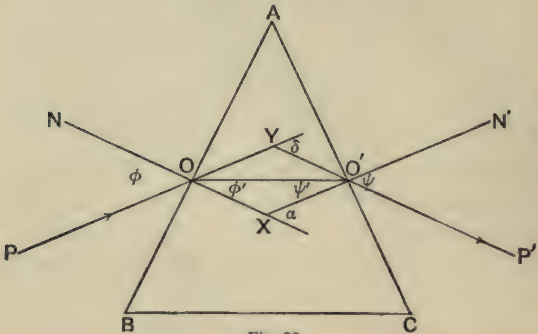


Fig. 26

therefore been turned through the angle δ, and the angle δ consequently measures the deviation.

$$
\begin{aligned}
\text{Let } \angle \text{PON} &= \phi \\
\angle \text{XOO}' &= \phi' \\
\angle \text{N}'\text{O}'\text{P}' &= \psi \\
\angle \text{OO}'\text{X} &= \psi'. \\
\text{Then } \angle \delta &= \angle \text{YOO}' + \angle \text{YO}'\text{O} \\
&= \phi - \phi' + \psi - \psi' \\
&= \phi + \psi - (\phi' + \psi') \\
\text{and } \phi' + \psi' &= \angle a.
\end{aligned}
$$

But since the angles of any quadrilateral figure are together equal to four right angles, the angles contained by the figure AOXO′ are together equal to four right angles. Now, since the angle XOA and the angle XO′A is each a right angle, the angle OAO′ and the angle OXO′ are together equal to

two right angles. Moreover, the angle O X O′ and the angle a are together equal to two right angles. Hence, the angle a is equal to the angle O A O′. Calling the angle O A O′ ϵ, we have—

$$\delta = \phi + \psi - \epsilon.$$

This angle δ varies in value, and its magnitude depends upon the size of the angle of incidence at the first surface. Its smallest value is obtained when the light traverses the prism in a direction at right angles to a line bisecting the apex angle of the prism. When this is the case $\phi = \psi$ and $\phi' = \psi'$, and the deviation is then said to be minimum. Calling this angle of minimum deviation Δ, we have—

$$\Delta = 2\phi - \epsilon.$$

$$\therefore \phi = \frac{\Delta + \epsilon}{2}.$$

$$\text{But } \sin\frac{(\Delta + \epsilon)}{2} = \mu \sin\phi'$$

$$= \mu \sin\left(\frac{\epsilon}{2}\right).$$

In this case ϕ' is equal to ψ' and $\phi' + \psi'$ has already been shown equal to ϵ. Consequently—

$$\mu = \frac{\sin\left(\frac{\Delta + \epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}.$$

To determine, therefore, the index of refraction

of the glass of which the prism is made, it is only necessary to measure its angle of minimum deviation and its apex angle. Both these measurements are readily made with the spectrometer. (See Appendix.)

We have already seen that the deviation is equal to $\phi + \psi - (\phi' + \psi')$ and that $\phi' + \psi' = \epsilon$, the apex angle of the prism. If the angles under consideration are all very small, the angles themselves, expressed in circular measure, may be written for their sines. Hence, instead of writing—

$$\sin \phi = \mu \sin \phi',$$

it is then legitimate to write—

$$\phi = \mu \phi' \text{ and } \psi = \mu \psi',$$

and we have—

$$\delta = \mu \phi' + \mu \psi' - \epsilon.^{1}$$
$$\therefore \delta = \mu \epsilon - \epsilon$$
$$= (\mu - 1)\epsilon.$$

[1] This formula is sufficiently accurate for the student of ophthalmology, and indeed is approximately true for all cases in which none of the angles is greater than 10°, for in this case the angles, expressed in circular measure, are very nearly equal to their sines.

The closeness of the approximation is shown by a comparison of the circular measure of one or two small angles with the natural sines of the same angles.

Circular measure of 2° = ·034897
Natural sine of 2° = ·034906
·02 per cent of error.

Circular measure of 5° = ·08726
Natural sine of 5° = ·08715
·12 per cent of error

Circular measure of 10° = ·17452
Natural sine of 10° = ·17364
·5 per cent of error.

Enumeration of Prisms.—The earlier manufacturers of ophthalmic prisms generally marked on each prism the number of degrees in its apex angle, and in many trial cases even now only such prisms are to be found. It is frequently stated in books that the deviation caused by a prism is equal to half the number of degrees in its apex angle. Thus a prism marked by this method number 10 would have a minimum deviation of 5°. The reason of this assumption is at once obvious on examination of the formula

$$\delta = (\mu - 1)\epsilon,$$

for if we take the value of μ for glass as 1·5, then

$$\mu - 1 = \tfrac{1}{2}.$$

Now the index of refraction for glass is not 1·5, but is generally considerably greater. Hence this method is not at all accurate. Of recent years a more rational system has prevailed, and prisms are now numbered according to their angles of minimum deviation. For any given prism the angle of minimum deviation may be expressed in the ordinary sexagesimal degrees, or in centradians, or in prism-dioptres. Nothing further need be said about the sexagesimal scale, but a few words of explanation are required as to the centradian and as to the prism-dioptre.

In an earlier part of this work (page 3) we have referred to the circular measurement of angles, and

there it was stated that the unit of this measurement is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle. This angle is approximately 57·295 degrees, and is called the radian. The centradian is the hundredth part of one radian, and is nearly equal to ·6 of a degree. Prisms can now be obtained marked in their angles of minimum deviation expressed in centradians.

The annexed diagram explains what is meant by a prism-dioptre.

Let A B (fig. 27) be a centimetre scale placed horizontally in some convenient position, such as

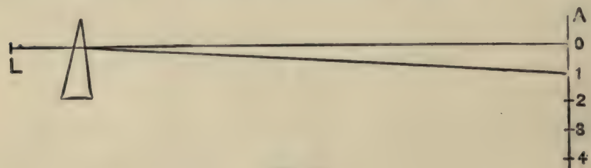


Fig. 27

along the wall of a room. Let L be a source of light with a slit in front of it, so arranged that when no prism is interposed an image of the slit is formed on the zero of the scale. Let a prism be inserted as in the diagram, at exactly 1 metre from the scale, and in such a position that the light passing through it undergoes minimum deviation. If now the image of the slit is found at 1 centimetre from zero, the glass is called a prism of minimum deviation of one prism-dioptre. If the

image be formed at 2 centimetres from the zero it is said to be of two prism-dioptres, and so on.

Uses of Prisms.—Amongst the various clinical purposes to which prisms are applied the following may be specified:—

(*a*) Where there is diplopia they form a handy and excellent method of measuring the angle between the axes of vision of the two eyes, *i.e.* the angle of squint.

(*b*) They are of service in measuring the range of convergence and in testing muscular defects.

(*c*) In many cases they must be used as therapeutic agents.

Measurement of the Angle of Squint by Prisms.—This can easily be done provided there is diplopia, although it must be confessed that the method by prisms possesses no advantages over that by means of tangent scales. It is further to be remembered that in many cases of squint there is under ordinary circumstances no diplopia. It, however, can generally be elicited on careful and repeated examination. Diagram 28 illustrates the method of measuring convergent strabismus. In the eye which does not appear to squint the image of the fixation object A is formed on the macula, while in the other eye it is formed not on the macula but at some point on the inner side of the retina and is projected by this eye to A′. Hence, as is well known, there is homonymous diplopia in

convergent strabismus. If now a prism be placed in front of the squinting eye, with its base outwards, it will cause the light coming to this eye to deviate outwards, and if the prism be of the proper strength to bring the image to the macula, binocular single vision for the fixation-object is once more restored. In making this experiment care must be taken that the prism is placed in front of the eye in as nearly as possible its position of minimum deviation relatively to the incident light. If the prism be marked, as all prisms ought to be, according to its angle of minimum deviation, then it at once measures the angle of the squint.

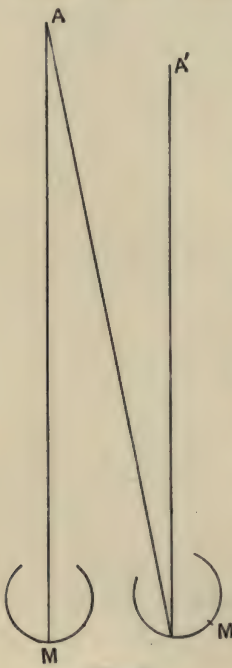


Fig. 28

One fallacy, however, may occur in making measurements which involve the external and internal recti, viz. that for lateral movements, under ordinary circumstances, the eyes have a great tendency to united action. Thus, generally speaking, in health no diplopia occurs when a prism of ten or fifteen degrees is placed in front of one of the eyes with

its base outwards, for under these circumstances there is immediately an impulse to convergence. In eyes which are not accustomed to binocular fixation, and in cases where the diplopia is elicited only with great difficulty, the measurements taken by the method indicated are substantially correct.

Where great accuracy is required, as in paresis of one of the muscles employed in lateral movements, the following method may be adopted. In front of one eye, preferably that which has the better vision, a prism of about two degrees minimum deviation is placed with its edge upwards and exactly horizontal. The effect of this is that one of the images is projected upwards. If the upper image is directly over the lower one there is no diplopia, at any rate for that point of fixation. If, on the other hand, the upper image is displaced laterally there is either homonymous or crossed diplopia according as the image is displaced towards the same side as the eye which has the prism in front of it, or to the opposite side. Homonymous diplopia corresponds with convergent strabismus and crossed diplopia with divergent. The amount of the strabismus can easily be measured by finding the prism which, with its base directly outwards or inwards, brings the upper image right above the lower, or which brings the lower image directly beneath the upper.

Measurement of the Range of Convergence with Prisms.—The range of convergence is usually

stated to consist of two parts, viz. a positive and a negative. The positive portion is easily understood. Let us suppose that when the eyes of the person who is being examined are at rest the axes of vision are parallel to each other. If then a suitable test object, such as an illuminated slit, be brought in the median line nearer and nearer to the eyes, there comes a point at which binocular vision is no longer possible, for the limit of the positive convergence has been exceeded. This point is called the *proximate point of convergence.* When both eyes are fixed for this proximate point the angle of convergence can very readily be ascertained, and may be expressed in ordinary sexagesimal measure or in metric angles of convergence.[1]

It can be measured by prisms in the following manner. Let the patient be placed at a considerable distance from the test object, which, in this case, should be a lighted candle. Printed letters are not suitable, because the effort of convergence necessitates an effort of accommodation which may render the letters so indistinct that the patient ceases to be able to read them. The patient hav-

[1] In ordinary ophthalmic practice angles of convergence are measured by the metric scale, which although not so accurate as the method just described, yet gives results sufficiently reliable for clinical work. In this method the intracorneal distance is entirely neglected, and whenever a man binocularly fixes a point a metre away, he is said to exercise one metric angle of convergence. If he fixes a point at half a metre, he is said to use two metric angles of convergence If the point be $\frac{1}{n}$ of a metre from him, n metric angles of convergence are required.

ing thus been placed at 20 feet from a lighted candle, a prism with its base directly outwards is placed in front of one eye. For a moment there may be diplopia, but, generally speaking, the instinctive effort of convergence which the patient makes soon re-establishes binocular fixation. If stronger and stronger prisms be used, one is at last obtained which the converging effort of the eyes cannot overcome. The diplopia remains permanent. The strongest prism, with its base outwards, which admits of binocular fixation, measures the near point of the positive convergence. In many cases it is a prism of ten, or even more, degrees of minimum deviation.

The negative part of the range of convergence depends upon the latent power of divergence. It is found by experiment that if in normal circumstances a weak prism be placed in front of one eye, with its base inwards, there is still single vision. Such a condition can only be brought about by the eye undergoing a slight movement of divergence. Generally speaking, a strong robust person still has single vision for distance when a prism of two or even three degrees of minimum deviation is placed with its base inwards before one eye.

CHAPTER III

LENSES

In a previous section we have studied refraction at a plane surface, and have seen that for a given substance the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction. In the present section we propose briefly to discuss the laws of refraction at spherical surfaces.

Every spherical surface may be regarded as composed of a number of small plane surfaces. The truth of this statement is illustrated by the following considerations. The surface of a small pond of water appears to the eye of the observer to be absolutely flat. Yet we know it is not so, but has a curvature with a radius equal to the distance between the surface of the pond and the centre of the earth. Still, the pond is so small compared with the total surface of the earth that it may truly be considered to have a plane surface. Now, if an extremely small portion of any curved surface be taken, it may be regarded as being a plane area.

Fig. 29 represents a ray of light Q N incident on the spherical refracting surface A H B, and suppose that Q N is parallel to O O′, the geometric axis of the figure. We propose, in the first place, to trace the direction of Q N after refraction. Let us suppose

that the medium in which QN is propagated is atmospheric air, and that the substance bounded by the spherical surface, of which AHB is a section, is glass. At the point N a portion of the sphere may be taken small enough to be regarded as a plane surface and coinciding with the tangent to the surface at the point N. A line drawn from the

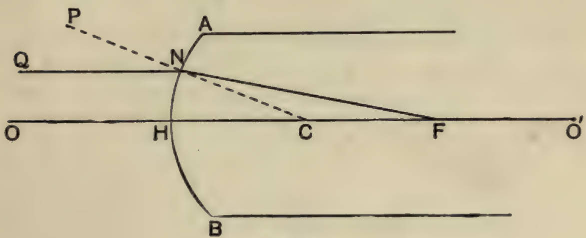


Fig. 29

point N to C, the centre of curvature, is the normal to the surface at N. Produce CN to P; QNP is the angle of incidence. As the second medium is the denser, QN, after refraction, is bent towards the perpendicular NC. QN, after refraction, will intersect the axis at some such point as F. The angle CNF is the angle of refraction, and we have sin QNP $= \mu$ sin CNF, where μ is the index of refraction of the glass. If the angles be all small—*i.e.* if they are, say, not greater than ten degrees or any less number—it is legitimate to write the angle expressed in circular measure for its sine. Hence, with this limitation, we have $i = \mu r$.

Now the angle PNQ is equal to the angle NCO,

and the angle NCO is equal to the angles CNF and NFC. Calling NCO i, and CNF r, the angle NFC is $i - r$. In the triangle CNF we have—

$$\frac{\sin r}{\sin\,(i-r)} = \frac{\mathrm{CF}}{\mathrm{R}},$$

where R is the radius of curvature. In the case which we are considering, viz. when all the angles are very small, we have—

$$\begin{aligned}
\frac{\mathrm{CF}}{\mathrm{R}} &= \frac{r}{(i-r)} \\
&= \frac{r}{\mu r - r} \\
&= \frac{1}{\mu - 1}. \\
\therefore \mathrm{CF} &= \frac{\mathrm{R}}{\mu - 1}. \\
\therefore \mathrm{HF} &= \frac{\mathrm{R}}{\mu - 1} + \mathrm{R} \\
&= \frac{\mu \mathrm{R}}{\mu - 1}.
\end{aligned}$$

Before refraction the ray QN was parallel to the axis OH; hence we are dealing with a pencil of parallel light, and the distance HF is defined to be the *posterior principal focal distance* of this surface.

In fig. 30 we have a ray of light QN, which before refraction is parallel to the axis O′O. In this case, after refraction, the light will be bent away from the perpendicular CN in some direction such as NF′, provided the second medium is optically less

dense than the first. The angle QNC is the angle of incidence $= i$, and PNF′ is the angle of refraction $= r$. Produce QN to K, then the angle QNC is the angle of incidence, and is equal to the angle NCH, and also to the angle PNK. Again, the angle

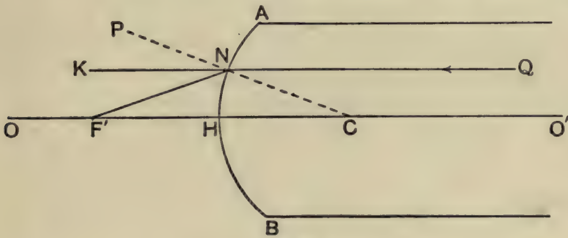


Fig. 30

HF′N is equal to the angle F′NK, *i.e.* to the angle F′NP minus the angle PNK. Thus in the triangle F′NC the angle NCF′ $= i$, and the angle NF′C $= r - i$, and NC = R (the radius of curvature),

$$\text{hence } \frac{\text{F}'\text{N}}{\text{R}} = \frac{\sin i}{\sin (r - i)}.$$

In the special case which we are considering, viz. where all the angles are very small,—

$$\frac{\text{F}'\text{N}}{\text{R}} = \frac{i}{r - i}$$

$$= \frac{i}{\mu i - i}$$

$$= \frac{1}{\mu - 1}.$$

$$\text{Hence F}'\text{N} = \frac{\text{R}}{\mu - 1}.$$

Now if the angle H F′ N be very small N F′ is very nearly equal H F′; and in the limit as N approaches H the two are equal to each other. As the light was parallel to the axis before refraction H F′ is defined to be the *anterior principal focal distance* of the surface, just as in the previous case H F was called the *posterior principal focal distance.*

Calling the posterior principal focal distance F_1 and the anterior principal focal distance F_2, we have—

$$F_1 = \frac{\mu R}{\mu - 1}$$

$$\text{and } F_2 = \frac{R}{\mu - 1}.$$

$$\therefore F_1 - F_2 = \frac{\mu R - R}{\mu - 1}$$

$$= \frac{R(\mu - 1)}{\mu - 1}$$

$$= R.$$

$$\text{Again, } \frac{F_1}{F_2} = \frac{\dfrac{\mu R}{\mu - 1}}{\dfrac{R}{\mu - 1}} = \mu.$$

Practical Examples.—Let the index of refraction of the lens in diagram No. 29 be 1·5, and the radius of curvature be 20 mm.; required the posterior principal focal length.

We have—

$$F_1 = \frac{1{\cdot}5 \times 20}{1{\cdot}5 - 1}$$

$$= 60 \text{ mm.}$$

With the same lens the anterior principal focal distance is $\frac{20}{\cdot 5} = 40$ mm. It will be observed from this that—

$$\begin{aligned} F_1 - F_2 &= 60 - 40 \\ &= 20 \\ &= R. \end{aligned}$$

The student must have observed that in the preceding paragraphs certain approximations have been made. The results obtained may be taken as representing without appreciable error the actual conditions present. It will be useful, however, to trace a few individual rays to their point of intersection with the axis, using for this investigation the law of Snell. In the first place, let parallel light be incident on the spherical surface at such a distance from the principal axis that the radius of curvature drawn to the point of incidence makes with the principal axis an angle of, say, five degrees. Let this radius of curvature be 20 mm. in length, and let the index of refraction be 1·5. We have already seen that for parallel light the angle of incidence at any point is equal to the angle which the radius drawn to that point makes with the principal axis. The angle of refraction r is easily obtained, for its sine is equal to $\frac{\sin i}{\mu}$ or $\frac{i}{\mu}$, since i is a small angle. The circular measure of 5° is, from tables, ·0873 radian.

$$\begin{aligned} \therefore r = \frac{i}{\mu} &= \frac{\cdot 0873}{1 \cdot 5} \\ &= \cdot 0582 \text{ radian} \\ &= 3^\circ \, 20', \\ \text{and } i - r &= 1^\circ \, 40' \\ &= 0 \cdot 291 \text{ radian.} \end{aligned}$$

In the triangle C N F we have—

$$\frac{\text{CF}}{\sin r} = \frac{\text{R}}{\sin (i - r)}; \ i.e. \ \frac{\text{CF}}{r} = \frac{\text{R}}{(i - r)}$$

$$\therefore \ \text{CF} = \frac{r\,\text{R}}{(i - r)} = \frac{\cdot 0582 \times 20}{\cdot 0291}$$

$$= 40 \text{ mm.},$$

by using the approximate formula. The more rigorous formula gives C F = 39·98—practically 40 mm. But C H = 20 mm. Hence the whole distance H F is equal to 59·98 mm. If light parallel to the axis is incident at a point on the surface of A B so that the angle N C H is equal to ten degrees, by calculation the line C F is found to be as nearly as may be also 40 mm. and, as before, H F is equal to about 60 mm. If, however, the angle

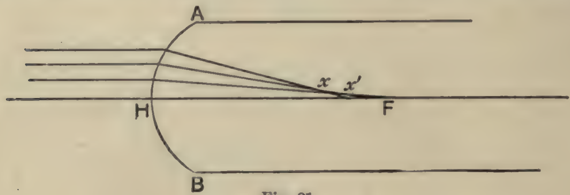


Fig. 31

N C H be taken equal to twenty degrees, C F is found to be approximately 38·4 mm. and H F is equal to 58·4 mm. Similarly, if the angle N C H be taken equal to thirty degrees, C F is found to be 36·4 mm. and H F is equal to 56·4 mm.

In the annexed diagram, fig. 31, lines are drawn corresponding to angles of ten degrees, twenty degrees, and thirty degrees. It will be observed that the farther the point of incidence is from the axis the nearer to the

surface A H B does the refracted ray cut the axis. In fact it is only the pencils of light which are very close to the axis that can be said to form a focus at the point F. It will be seen also that the various rays, after refraction, cut each other at such points as x and x'. The curve drawn through all such points forms what is called a caustic curve, and the whole of the phenomena are generally classed under spherical aberration.

One or two facts are easily established from a consideration of fig. 32, provided certain approximations are made. It has already been shown that

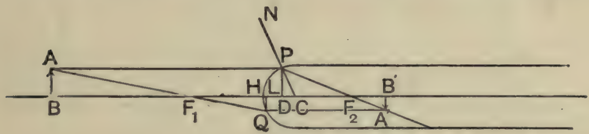


Fig. 32

if parallel light travelling in the direction AP or BH is incident on the spherical surface P H Q, then it is approximately brought to a focus at the point F_2; and conversely, if parallel light is incident on the surface P H Q in the direction of B′H, then it is brought to a focus at some such point as F_1. In both cases it is assumed that the light travelling between B and H is in a medium of lesser density than between H B′.

These facts give a well-known method of geometrically finding the image of any object situated as A B. Let us suppose that from every point of its surface rays proceed in all possible directions. Then one ray must be the direction A P, parallel to the

axis B H. As this light is parallel to the axis before incidence, after refraction it passes through the posterior focus F_2, and the direction of the ray is P A′. Another ray must be in the direction A F_1, which is incident at the point Q. As this ray passes through the anterior principal focus, after refraction it is parallel to the axis. The point A′, where these two rays intersect, is the image of A. By a similar construction we can determine the image of every point in A B, and thus it can be proved that A′B′ is the image of A B.

The object and image are situated at foci conjugate to each other. If the object is at A B its image is at A′B′, and, conversely, if the object were at A′B′, its image would be at A B. Further, in this particular case the image is a real one—rays coming from A actually meet at A′—and, like all real images, it is inverted.

The following results are of great importance in the theory of lenses and should be carefully studied.

In fig. 32, A B, the object, is equal to P D. Let each of these be of length o. Similarly, A′B′, the image, is equal to Q L, either of which may be written i.

Further, let

$$
\begin{aligned}
\mathrm{BH} &= f_1, \\
\mathrm{B'H} &= f_2, \\
\mathrm{HF_1} &= \mathrm{F_1}, \\
\mathrm{HF_2} &= \mathrm{F_2}, \\
\mathrm{BF_1} &= l_1, \\
\mathrm{BF_2} &= l_2.
\end{aligned}
$$

Then, in the similar triangles, QLF_1 and ABF_1, we have—

$$\frac{QL}{AB} = \frac{LF_1}{F_1B} = \frac{i}{o} \ldots\ldots\ldots\ldots\ldots (1)$$

Again, from the similar triangles, $A'B'F_2$ and PDF_2, we have—

$$\frac{A'B'}{DP} = \frac{B'F_2}{DF_2} = \frac{i}{o} \ldots\ldots\ldots\ldots\ldots (2)$$

Hence—

$$\frac{LF_1}{F_1B} = \frac{B'F_2}{DF_2} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

But BF_1 is equal to l_1, and, if we neglect the small distance HL, LF_1 is equal to F_1. Also, $B'F_2$ is equal to l_2, and DF_2, if we neglect the distance DH, is equal to F_2. Making these substitutions, we find that—

$$\frac{F_1}{l_1} = \frac{l_2}{F_2}.$$

$$\text{Hence } l_1\, l_2 = F_1\, F_2 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\text{But } l_1 = f_1 - F_1$$

$$\text{and } l_2 = f_2 - F_2.$$

$$\text{Hence } (f_1 - F_1)(f_2 - F_2) = F_1\, F_2 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Multiplying out and dividing by $f_1\, f_2$, we find that—

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1.$$

In the above we have made the usual approximations, which, however, cause no appreciable error for the case under consideration.

As already pointed out, the image A′ B′ is a real image, for the rays of light do actually meet. Moreover, a real image is formed so long as the object is at a greater distance in front of the refracting surface than the anterior principal focus. It will be observed from the diagram that the image is smaller than the object. Had the object A B been placed at twice the principal focal distance in front of the

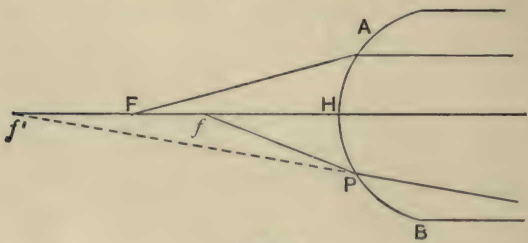


Fig. 33

surface, then B F_1 would have been equal to F_1 L and the triangles would have been not merely similar but equal in every respect. The image and object would then have been equal to one another. Had the object been at a less distance than twice the principal focal length, but at a greater distance than the principal focal length, the image would have been greater than the object. If the object is between the principal focus and the refracting surface, then the rays of light after refraction continue to diverge. This is shown in diagram No. 33, where the object is placed at *f* in front of the refracting surface A H B, whose principal focal length

is H F. Here, after refraction, the ray f P appears as if coming along the dotted line from f', which is therefore the *virtual* conjugate focus of f.

Hitherto we have considered the conditions of refraction at a single spherical surface; we must now turn our attention to such lenses as are bounded by two spherical surfaces. In these we must consider the refraction at both surfaces. In ophthalmic work the forms of lenses ordinarily employed are the biconvex, the plano-convex, and the meniscus-convex. We have also the biconcave, the plano-concave, and the meniscus-concave. Convex glasses, of whatever kind, tend to make rays of light converge, while, on the other hand, concave lenses tend to make them diverge. Convex lenses have this feature in common, that they are thicker at the centre than at the periphery. Concave lenses, on the other hand, are thinnest at the centre. In fig. 35 we have in section an example of each.

We will, in the first place, consider biconvex lenses. In such glasses frequently both surfaces have equal radii of curvature. They are generally divided into two classes. In the first group are included those which are so thin that their thickness does not require to be taken into account, as it is very small compared with their focal lengths. Ordinary spectacle glasses are for the most part examples of this class. We shall speak of them hereafter as *thin lenses*. In the second class are lenses whose thickness cannot be neglected rela-

tively to their focal lengths. These we shall call *thick lenses.*

In fig. 34 A B is a thin lens, and light is incident on the surface A H_1B, which is generally spoken of as the first surface. After refraction at that surface it is incident on A H_2 B, which is generally called the second surface. The centre of curvature of the first surface is C_1, and its radius of curvature is R_1. The centre of curvature of the second surface is C_2, and its radius of curvature is R_2. The right line joining the centres of curvature is defined as the axis of the lens. The point P, at which rays of light parallel to the axis before incidence are brought to a focus, is called the *principal focus,* and the distance between the principal focus and a thin lens is called its *focal length.*

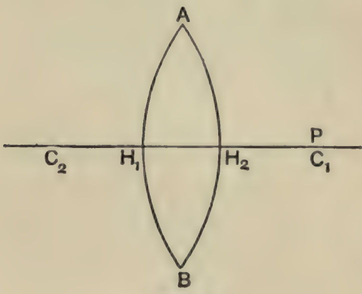
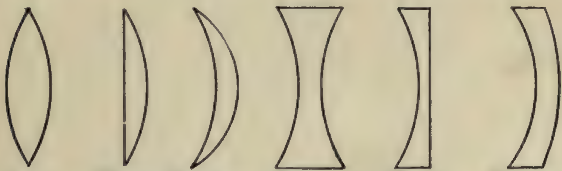


Fig. 34

To find a Formula for the Focal Length of a Thin Biconvex Lens.—Rays of light which are incident on the first surface are, after refraction, directed towards a point F, the distance of which from the surface of the lens can be obtained by the formula on page 62, and is equal to $\frac{\mu R_1}{\mu - 1}$.

As the lens is thin its thickness may be disregarded without any appreciable error, and the rays of light are incident on the second surface as if directed towards this point F; and the distance of the object from the second surface is equal to $\frac{\mu R_1}{\mu - 1}$. Moreover, as these rays of light are converging, the sign of this quantity is negative. Now if we take into consideration the second surface alone, rays of



Biconvex Plano-convex Meniscus-convex Biconcave Plano-concave Meniscus-concave

Fig. 35

light parallel to the axis entering the lens from the surrounding medium are bent towards a focus F_2, whose distance from the surface is $\frac{\mu R_2}{\mu - 1}$. Further, rays of light, which in the lens are parallel to the axis, on emerging into the air are bent to a point F_1, whose distance from the lens is $\frac{R_2}{\mu - 1}$. Applying to the second surface the formula we have already obtained,—

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1,$$

we have by substitution for F_1, F_2, and f_2:—

$$\frac{\dfrac{R_2}{\mu - 1}}{f_1} + \frac{\dfrac{\mu R_2}{\mu - 1}}{\dfrac{-\mu R_1}{\mu - 1}} = 1,$$

$$\text{that is } \frac{1}{f_1}\frac{R_2}{\mu - 1} - \frac{R_2}{R_1} = 1.$$

$$\therefore \frac{1}{f_1(\mu - 1)} - \frac{1}{R_1} = \frac{1}{R_2},$$

$$\text{or } \frac{1}{f_1} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

f_1 thus gives the position of the conjugate focus obtained by refraction for the second surface. But as the light originally was a pencil parallel to the axis of the lens before it encountered the first surface, the distance between the lens and f_1 is the principal focal length of the lens itself. Hence, if the focal length of any thin biconvex lens be F, we have—

$$\frac{1}{F} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

If each surface has the same radius of curvature, we may write the formula—

$$\frac{1}{F} = (\mu - 1)\frac{2}{R}.$$

Further, if in this last case the index of refraction μ be equal to 1·5, as it very approximately is for some kinds of glass, then the focal length is equal to the radius of curvature of either surface.

Hence for any thin glass lens of this kind, the radius of curvature of each face is roughly equal to the focal length.

Plano-Convex Lens.—Little difficulty is experienced in deducing the formula for the plano-convex lens, for in this case either R_1 or R_2 is equal to infinity, and—

$$\frac{1}{\infty} = 0,$$

so for the plano-convex lens—

$$\frac{1}{F} = (\mu - 1)\frac{1}{R}.$$

Convex-Meniscus Lens.—Let light parallel to the axis be incident on the convex surface. It is refracted towards a point whose distance from that surface is equal to $\frac{\mu R_1}{\mu - 1}$. This distance with a negative sign is represented by f_2 in the formula—

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1,$$

where F_1 and F_2 are respectively the first and second principal focal lengths for refraction at the second surface.

Substituting in the above equation—

$$-\frac{\dfrac{R_2}{\mu - 1}}{f_1} + \frac{\dfrac{\mu R_2}{\mu - 1}}{\dfrac{-R_1}{\mu - 1}} = 1.$$

$$\therefore \frac{R_2}{(\mu - 1)f_1} = \frac{R_2 - R_1}{R_1}.$$

$$\therefore \frac{1}{f_1} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

As the light was parallel before incidence at the first surface, f_1 is in reality the principal focal length of this convergent meniscus.

Biconcave Lens.—Similar demonstrations are applicable to diverging lenses.

In fig. 36 a pencil of parallel light is incident on one surface. Here it undergoes refraction, and

Fig. 36

after refraction the rays diverge as if coming from the point P, which is consequently a virtual focus, situated on the same side of the lens as the incident light. This virtual focus is therefore the object for the refraction at the second surface. Its distance from the first surface is, as has been shown,—

$$\frac{\mu R_1}{\mu - 1},$$

where R_1 is the radius of curvature of the first surface. At the second surface the focus for light

parallel before incidence is obtained by the formula—

$$F_2 = -\frac{\mu R_2}{\mu - 1},$$

and the focus for light which is parallel in the substance of the lens incident on the second surface is found by the formula—

$$F_1 = -\frac{R_2}{\mu - 1}.$$

Substituting these values in the formula—

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1,$$

we have—

$$\frac{-\dfrac{R_2}{\mu - 1}}{f_1} + \frac{-\dfrac{\mu R_2}{\mu - 1}}{+\dfrac{\mu R_1}{\mu - 1}} = 1,$$

$$\therefore -\frac{1}{f_1} = (\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Here the value of f_1 is negative, which means that it is on the same side of the lens as that on which the parallel light was originally incident. It is, therefore, the principal focal length of the entire lens, and thus we derive the standard formula for the biconcave lens—

$$\frac{1}{F} = -(\mu - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Plano-Concave Lens.—In the plano-concave lens

the conditions are very simple, for one radius is infinitely great. Hence its formula is—

$$\frac{1}{F} = -(\mu - 1)\frac{1}{R},$$

where R is the radius of the curved surface.

Divergent Meniscus.—The formula for this lens is—

$$\frac{1}{F} = -(\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

The Formation of an Image by a Thin Biconvex Lens.—A B (fig. 37) is a thin biconvex lens. One of its principal foci is at F_1, the other at F_2.

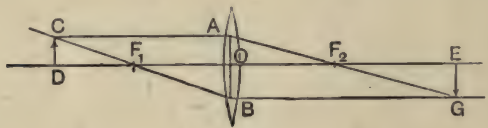


Fig. 37

Let C D be an object placed in front of the lens at a greater distance than the principal focal length. To find the image of this object we see that from the point C rays of light diverge in all possible directions. One ray C A is parallel to the axis D O; after refraction it passes through the focal point F_2 and proceeds towards the point G. Another ray of light from C passes through the focal point F_1, and after refraction is parallel to the axis. The point G, at which these two rays meet, is the image of C.

In the same manner an image can be obtained

for every point in C D, and these would be found collectively to form the complete image G E.

Let the distance $F_1O = F_2O$ be called F, the distance $DF_1 = l_1$, and $F_2E = l_2$. We have $OB = EG$ and $CD = AO$, therefore from the triangles CDF_1 and BOF_1 we have—

$$\frac{\text{object}}{\text{image}} = \frac{l_1}{F}.$$

Again, in the triangles AOF_2 and GEF_2, we have—

$$\frac{\text{object}}{\text{image}} = \frac{F}{l_2}.$$

$$\text{Hence } l_1 l_2 = F^2.$$

Let the distance D O be equal to f_1 and the distance O E equal to f_2. Then $l_1 = f_1 - F$ and $l_2 = f_2 - F$. Substituting in the above equation we have—

$$(f_1 - F)(f_2 - F) = F^2,$$

$$\therefore f_2 F + f_1 F = f_1 f_2.$$

Dividing by $f_1 f_2$ we have—

$$\frac{F}{f_1} + \frac{F}{f_2} = 1.$$

When the object is in front of the lens, at exactly twice the focal distance, then the image is of the same size as the object. If the object be at a greater distance than twice the focal length, then the image is smaller than the object. Moreover,

the farther away the object is situated the nearer is the image to the second principal focus. When the object is at an infinite distance, in other words, when the rays of light are parallel, the image is situated at the principal focus. Conversely, when the object is at a greater distance than the principal focus, but not at so great a distance as twice the principal focus, the image is larger than the object.

Only one other case requires consideration, viz. when the object is (in front of the lens) at a

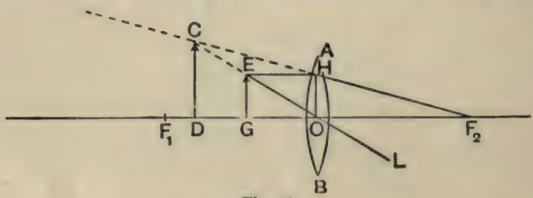


Fig. 38

shorter distance than the principal focal length. Let A B (fig. 38) be the lens and E G the object, a ray of light E H, parallel to the axis, after refraction is directed towards the principal focus F_2. Another ray E O passing through O where the thin lens cuts the axis is not refracted at all, and hence forms the straight line E O L. It will be observed from the diagram that these two rays of light after passing through the lens are still divergent, and therefore do not meet on the same side of the lens as F_2. If produced backwards, however, they meet at the point C; thus, if an eye is so placed on the same side of the lens as F_2 that it receives these two rays

of light, it will see the image of E at the point C, which is therefore the virtual image of E, and C D is the virtual image of E G. By a similar construction and by simple experiment it can be shown that the closer the lens is to the object the smaller is the virtual image; thus, if a strong convex lens be placed on a line of print the letters seem to be scarcely magnified at all. If now the lens be gradually removed from the line towards the observer, the letters will appear to be more and more magnified. This is the principle of the simple microscope. If the observer continues to bring the lens nearer to him, a point is reached at which no distinct image is obtained. Here the object is approximately at the principal focus of the lens. On drawing the lens still nearer to him the observer will see, if the lens be strong, an inverted image of the type. This latter is a real image, and therefore an inverted one. The former was a virtual image and erect.

Formation of an Image by a Thin Concave Lens.—Let A B (fig. 39) be an object in front of a concave lens of which F and F_1 are the principal focal points. Rays of light are sent from A B in all possible directions. One ray of light A H is parallel to the axis B F F_1 of the lens. After refraction it leaves the lens as if proceeding from the principal focus F, and its ultimate direction is therefore H K. Another ray A C forms a *secondary* axis (see p. 83), and is therefore not refracted at all; it continues in the direction A C L. Consequently the

point at which the ray H K, when produced backwards, intersects A C L is the *virtual image* of the point A. In this way D G can be shown to be the virtual image of A B. From this and the preceding paragraph it will be observed that the virtual image formed by a convex lens is larger than the object,

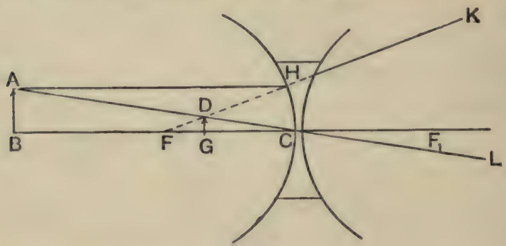


Fig. 39

while in the case of a concave lens it is smaller. The truth of this fact is at once apparent when the student looks at any object through a concave lens. The object looked at appears to be smaller than it is in reality.

When a patient is being corrected for shortsightedness he often states that with the glass in front of his eye the object at which he is looking appears to be diminished in size. When this occurs it is generally due to the fact that the lens which at the time is being used is too concave, and that a weaker one must be employed.

Thick Lenses. — In studying refraction at a single surface we came upon several points which were found to be of great importance, and by means

of which a geometrical construction for the image of an object was easily obtained. These were: (*a*) the anterior and posterior focal points, (*b*) the point on the refracting surface at which the ray of light enters the second medium, termed the principal point, (*c*) the centre of curvature, termed the nodal point.

The peculiarity of the nodal point is, that all rays of light which, while external to the surface, are directed towards this point, pass into the surface normally, and consequently are not refracted. Every line drawn from the nodal point is an axis. The one which also passes through the geometric centre of the refracting surface is termed the *principal axis*, the others are secondary axes.

The problem, however, becomes more difficult where we have to take into consideration an optical system composed of several surfaces separated from each other by media which have different indices of refraction. Theoretically it is possible to trace the path of a pencil of light through such a complicated system, for the image formed by the first refracting surface is the object for the second, and the image formed by the second is the object for the third, and so on; but while theoretically possible, in practice it is extremely difficult to carry out the construction with any degree of accuracy. Thanks, however, to the work of Gauss, a simple method has been found of solving the problem with sufficient accuracy for all practical purposes. The conditions to which it applies are that all the lenses

composing the system are so centred that their principal axes may be regarded as coinciding with each other, and that only a small portion of each refracting surface surrounding the principal axis is considered. In such a system Gauss and his successors have demonstrated that there are two focal points, two principal points, and two nodal points. These are called the six cardinal points.

The focal points present no difficulty. Light which is parallel to the axis before entering the system in one direction is brought to a focus at one of the focal points. Also, light parallel to the axis before entering the system on the other side is focussed at the other principal focal point. Hence it is common to speak of the anterior and posterior principal foci. Thus, for example, in the human eye we have several refracting surfaces, viz. the anterior and posterior surfaces of the cornea, the anterior and posterior surfaces of the lens, while the different layers of the lens within its capsule afford other refracting surfaces. Moreover we have several refracting media, such as the corneal tissue, the aqueous humour, the lens substance, the vitreous humour. Now, if a pencil of light parallel to the optic axis be incident on the cornea, it will be brought to a focus at the posterior focal point of the eye. If, on the other hand, light parallel to the axis leave the retina proceeding towards the cornea, it will be brought to a focus at the anterior focal point.

In addition to the six cardinal points which we have just mentioned, we have four planes, namely, two focal planes and two principal planes. A plane through the anterior focal point, at right angles to the optic axis, is the anterior focal plane, and a similar plane through the posterior focal point is the posterior focal plane. The principal planes, generally called the first and second principal planes, are planes respectively passing through the anterior and posterior principal points at right angles to the optic axis.

A few words of general explanation as to the optical properties of these points and planes will be of use to the student. The nodal points are, as defined by Gauss, two points situated on the principal axis of the system, and have this peculiarity, that if a ray of light before the first refraction is directed towards the first nodal point, it appears after the last refraction to have come from the second nodal point, and its path from the second is parallel to its original direction. Thus the direction is not changed; the effect of refraction is merely to cause a certain lateral displacement. If the incident light coincides with the geometric axis, this displacement does not take place, for both points are in line with the incident light.

If, however, a ray of light is directed to the first nodal point before refraction, so as to make an angle with the principal axis, after the last refraction it appears as if coming from the second nodal point

parallel to its original direction and consequently making an equal angle with the principal axis. The one nodal point is the image of the other.

Principal Planes.—These are two in number, and are at right angles to the principal axis. They have this peculiarity, that a ray of light which before the first refraction is directed towards a point in the first principal plane, after the last refraction appears as if coming from the corresponding point in the second. The points at which the principal planes are cut by such a ray are on the same side of, and at the same distance from, the principal axis. Hence the second principal plane is the image of the first, and object and image are of the same size and have the same direction.

Focal Planes.—In such a system as we are describing, viz. one composed of different refracting media and several surfaces all centred on the same axis, we have seen (p. 84) that there are, for rays near the axis, two principal focal points. Through each a plane may be drawn at right angles to the axis. These are the focal planes. If a ray of light come from a point in a focal plane other than that through which the principal axis passes, and is directed to the first nodal point, after the last refraction it appears to come from the second nodal point and is parallel to its original direction.

In the annexed diagram (fig. 40), if the light travels from left to right, F_1 is the anterior principal focus of the system, F_2 is the posterior. $F_1 H_1$

is the anterior focal length and $F_2 H_2$ is the posterior focal length. $H_1 H_2$ are respectively the first and second principal points, and $K_1 K_2$ the first and second nodal points. $F_1 H_1$ is equal to $K_2 F_2$, and $F_2 H_2$ is equal to $F_1 K_1$. Therefore $F_2 H_2 - F_1 H_1$ is equal to $F_1 K_1 - F_2 K_2$; *i.e.* the distance $H_1 H_2$ is equal to $K_1 K_2$, and either of these distances is equal to the difference between the focal lengths.

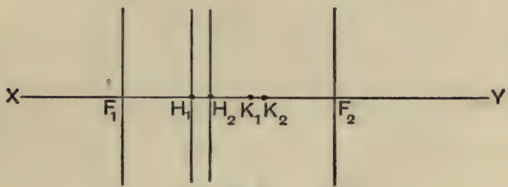


Fig. 40

Lastly, the two principal focal distances are to each other in the ratio of the indices of refraction of the first and last media. Calling the first focal distance F, and the second focal distance F′, and the index of the first medium, which, generally speaking, is atmospheric air, μ_1 and that of the last medium μ_2, we have—

$$\frac{F}{F'} = \frac{\mu_1}{\mu_2}.$$

In most optical instruments, such as a thick lens, although not in the eye, the first and last media are the same, viz. the surrounding atmosphere. In this case $\mu_1 = \mu_2$, and consequently $F = F'$, and thus the nodal points and principal points coincide.

In fig. 41 we have illustrated the six cardinal points and the two principal planes, and by means of these we can find the image of the object A B. The image of the point A is obtained as follows:—

One ray of light from A must be directed towards the point K_1; the ultimate direction of this is parallel to A K_1, and appears as if coming from K_2. Another ray must, before refraction, be parallel to

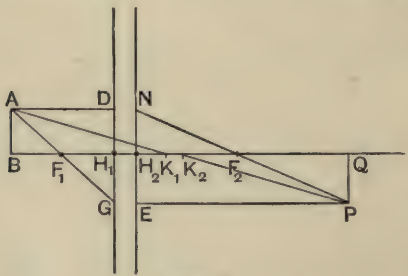


Fig. 41

the axis. It is directed towards the first principal plane at the point D. After refraction it appears as if coming from the similar point N on the second principal plane. Moreover, as it was parallel to the axis before refraction, it passes through F_2, the posterior focal point. The two points N and F_2 determine the direction of this ray, *i.e.* N F_2P. The point P, at which these two lines intersect, is the image of A, and hence it can be shown that P Q is the image of A B.

The point P could also have been found in another way. One of the rays from A must pass

through the anterior principal focus F_1, and is directed towards the point G in the first principal plane. After refraction it is parallel in direction to the axis B Q, and appears to come from the point E on the second principal plane, $E H_2$ being equal to $G H_1$.

From the similar triangles $A B F_1$ and $G H_1 F_1$,

$$O : I = l . F,$$

and from the triangles $N H_2 F_2$, and $P Q F_2$,

$$O : I = F' : l'.$$

Hence as before (page 73)—

$$\frac{F}{f} + \frac{F'}{f'} = 1.$$

The Thick Biconvex Lens.—We have already found certain formulæ with which the conjugate foci of a lens can readily be determined on the assumption that the thickness of the lens may be neglected. For the most part they are applicable to all the lenses used in ophthalmic practice, their thickness being inconsiderable as compared with their focal lengths. In one or two of the higher numbers, however, this assumption leads to error, and we have thought it advisable to add a special section on thick lenses.

Fig. 42 represents a thick lens of which one surface is B S′ and the other D S. C′ is the centre of curvature of the surface B S′ and C of D S.

Draw any radius C D of the surface D S, and from C′ draw the radius C′ B parallel to C D. Join B D. Thus the angle C D O is equal to the angle C′ B O. If now O be considered as a luminous point, sending light in the directions O D and O B, the angle of incidence O D C is equal to the angle of incidence O B C′, and consequently the angle of refraction

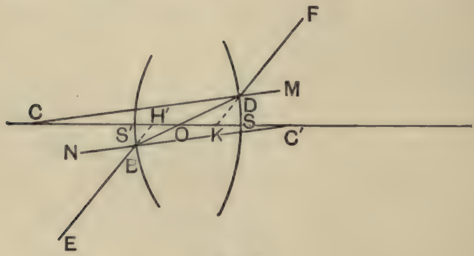


Fig. 42

F D M is equal to the angle of refraction E B N. Consequently F D is parallel to B E. Therefore, if light come from F to D, its ultimate direction after refraction is B E. Thus the direction of the emergent light is parallel to that of the incident. Hence the point K, the point on the axis to which the incident light is directed, and H′, the point on the axis from which the refracted light appears to come, are nodal points.

The point O is defined as the optical centre. From the similarity of the triangles B O C′ and D O C, and of the triangles S′ O B and S O D, we have—

$$C'B : C'O :: CD : CO \quad \therefore \; C'B : S'O :: CD : SO.$$

Thus the optical centre divides the thickness of the lens into two parts, which are proportional to the radii of curvature of the surfaces.

Calling C′B r', and CD r'', we have—

$$\frac{\text{S}'\text{O}}{\text{S O}} = \frac{r'}{r''}$$

Let fig. 43 represent a thick biconvex lens whose surfaces are S′ and S″, distinguished respectively as the first and second surface. We may call—

the thickness of the lens ... ... $e = a + b$;

the radius of curvature of the first surface ... ... r';

the radius of curvature of the second surface ... ... r'';

the first focal distance of the first surface, $\text{S}_1' f_1'$ ... ... f_1';

the second focal distance of the first surface, $\text{S}' f_1''$ ... ... f_1'';

the first focal distance of the second surface, $\text{S}'' f_2'$ ... ... f_2';

the second focal distance of the second surface, $\text{S}'' f_2''$ ... ... f_2'';

the index of refraction of the surrounding medium... ... ... μ';

the index of refraction of the glass of which the lens is made ... μ''.

If we consider the surfaces separately, we have, for the first surface,—

$$f_1' = \frac{\mu' r'}{\mu'' - \mu'},$$

$$f_1'' = \frac{\mu'' r'}{\mu'' - \mu'}$$

and for the second surface—

$$f_2' = \frac{\mu'' r''}{\mu'' - \mu'}$$

$$f_2'' = \frac{\mu' r''}{\mu'' - r'}$$

Hence—

$$\frac{f_1''}{f_2'} = \frac{r'}{r''}$$

that is, the second focal distance of the first surface and the first focal distance of the second surface are to each other as the radii of curvature.

To find the Principal Planes of a Biconvex Lens.—According to Gauss's definition (see p. 86), a ray which is directed to a point in the first principal plane before refraction appears, after refraction, to come from the corresponding point in the second plane. The one is the virtual image of the other, and both images are of the same size, and situated on the same side of the principal axis.

The following is an easy geometric method of finding the principal planes.

Let Q′P′ (fig. 43) be a ray of light parallel to the axis and incident on the surface S′ at the point

P′. After refraction it will be directed to the point f_1''. Also let Q″P″ be a ray of light in the same straight line as Q′P′, *i.e.* parallel to the axis, and at the same distance from it as Q′P′, and let Q″P″ be incident on the surface S″, at the point P″. After refraction it is directed to the point f_2'. These two refracted rays intersect at M; from M let fall a perpendicular MM′ on the axis. Let us now suppose that MM′ is a luminous plane seen in section. One ray of light, MP′, emitted by the plane is, after refraction, parallel to the axis. Another, MN′, is before refraction parallel with the axis, and therefore after refraction it is directed to f_1'. These two rays, if produced backwards, meet at the point H′, which is therefore the virtual image of M, and H′H′ is the virtual image of MM′. Similarly it can be shown that H″H″ is a virtual image of MM′. But these virtual

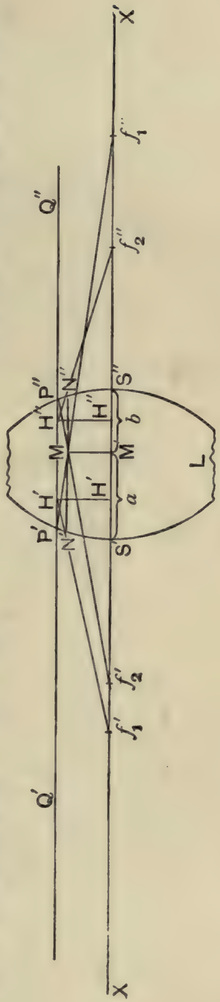


Fig. 43

images are of the same size, for $Q' Q''$ is parallel to $x x'$, and they are on the same side of the axis; they are therefore, according to definition, the principal planes.

The position of $M M'$ is easily found. From the triangles $P' S' f_1''$ and $M M' f_1''$ we have, $P' S'$ being small,

$$\frac{P' S'}{M M'} = \frac{S' f_1''}{M' f_1''} = \frac{f_1''}{f_1'' - a}.$$

Again, from the triangles $P'' S'' f_2'$ and $M M' f_2'$ we have—

$$\frac{P'' S''}{M M'} = \frac{S'' f_2'}{M' f_2'} = \frac{f_2'}{f_2' - b}.$$

But since $P' S' = H' H'$, and $P'' S'' = H'' H''$, then $P' S' = P'' S''$, and—

$$\frac{f_1''}{f_1'' - a} = \frac{f_2'}{f_2' - b}.$$

In this case—

$$\frac{a}{b} = \frac{f_1''}{f_2'} = \frac{r'}{r''}.$$

Therefore, to find the point M', the thickness of the lens must be divided proportionately to the radii of curvature. It is nearer the surface with the shorter radius of curvature.

From the last formula we have—

$$\frac{a}{f_1''} = \frac{b}{f_2'} = \frac{e}{f_1'' + f_2'}$$

$$\therefore a = \frac{e f_1''}{f_1'' + f_2'} \text{ and } b = \frac{e f_2'}{f_1'' + f_2'}.$$

We have now only to determine the position of the points H' and H'' in the principal axis. Let $S'H' = h'$ and $S''H'' = h''$. From the similar triangles $f_1'H'H'$ and $f_1'S'N'$ we have—

$$\frac{H'H'}{N'S'} = \frac{P'S'}{N'S'} = \frac{f_1'H'}{f_1'S} = \frac{f_1' + h'}{f_1'}.$$

Also from the triangles $P'S'f_1''$ and $MM'f_1''$ we have—

$$\frac{P'S'}{MM'} = \frac{P'S'}{N'S'} = \frac{S'f_1''}{M'f_1''} = \frac{f_1''}{f_1'' - a}.$$

$$\therefore \frac{f_1' + h'}{f_1'} = \frac{f_1''}{f_1'' - a}, \text{ since each } = \frac{P'S'}{N'S'}$$

substituting the value of a obtained above we have—

$$h' = \frac{e f_1'}{f_1'' + f_2' - e}$$
$$= \frac{e \mu' r'}{\mu'' r' + \mu'' r'' - e(\mu'' - \mu')}.$$

Similarly it can be shown that—

$$h'' = \frac{e f_2''}{f_1'' + f_2' - e}$$
$$= \frac{e \mu' r''}{\mu'' r' + \mu'' r'' - e(\mu'' - \mu')}.$$

The Principal Focus of a Thick Biconvex Lens.—Parallel rays of light incident on the surface of a biconvex lens are brought to a focus at a point. This statement is only true for rays situated near the axis of the incident pencil. At

present, however, we are dealing only with such rays. Let Q x', a ray parallel with the axis of the lens L (fig. 44), be incident on its surface at the point x'. The focus of this surface, considered by itself, is the point Q$'$. The incident light is therefore refracted in the direction x' Q$'$. At y' it, however, is refracted again, and our object is to

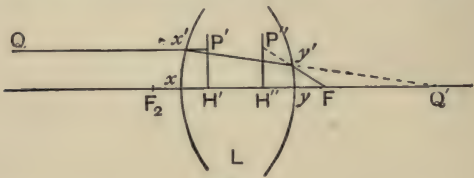


Fig. 44

find the point at which the ray, after this second refraction, intersects the principal axis. The point at which it does so is the principal focus. Before the first refraction the ray Q x' was directed to P$'$ in the first principal plane. After the last refraction it must appear as if coming from the corresponding point P$''$, in the second principal plane. We have thus determined two points, P$''$ and y', in the course of the refracted ray produced backwards. Hence its ultimate direction is P$''$ y' F; F is the principal focus and H$''$ F is the principal focal length.

From the similar triangles $x' x$ Q$'$ and $y' y$ Q$'$ we have—

$$\frac{x\,Q'}{y\,Q'} = \frac{x\,x'}{y\,y'}$$

Again, from the triangles $P'' H'' F$ and $y' y F$, we have—

$$\frac{H'' F}{y F} = \frac{P'' H''}{y y}.$$

But when dealing with a small arc of the curved surface, as in this case, $P'' H'' = x' x$. Hence—

$$\frac{x Q'}{y Q'} = \frac{H'' F}{y F}.$$

Calling $x Q'$, f_1'', and $H'' F$, F'', we have—

$$\frac{f_1''}{y Q'} = \frac{F''}{y F}.$$

But $y F = F'' - h''$ and $y Q' = f_1'' - e$. Therefore—

$$\frac{f_1''}{f_1'' - e} = \frac{F''}{F'' - h''}$$

Substituting for h'' its value obtained on page 95, we have—

$$F'' = \frac{f_1'' f_2''}{f_1'' + f'_2 - e}.$$

Similarly it can be shown that—

$$F' = \frac{f_1' f_2'}{f_1'' + f_2' - e}.$$

But it has already been shown (page 94) that for a biconvex lens—

$$\frac{r'}{r''} = \frac{f_1''}{f_2'} = \frac{f_1'}{f_2''} \text{ i.e. } f_1'' f_2'' = f_1' f_2'.$$

Therefore $F' = F''$.

The conjugate foci of a thick biconvex lens are easily obtained. In fig. 45 let A B be an object in front of the lens L. From the point A a ray A P′, parallel to the axis, is incident on the lens. After refraction it appears to come from the corresponding point P″ on the second principal plane, and passes through the focal point F′. The ray A F, directed to the point H′ in the first principal plane, appears, after refraction, to come from the point H′ below K″, and is parallel to the axis. The point A′ at which these two rays meet is therefore the image of A, and similarly it can be shown that A′ B′ is the image of A B.

Calling—

the distance of the object from the first principal plane, B K′, ... f;

the distance of the image from the second principal plane, B′ K″, ... f';

the distance of the object from the first focus, B F, ... l';

the distance of the image from the second focus, B′ F′, ... l'';

the principal focal distance, F K′ = F′ K″, F;

also A B = P″ K″ ... ... O;

and K′ H′ = B′ A′ ... ... i;

we have—

$$\frac{\text{O}}{i} = \frac{l'}{\text{F}} = \frac{\text{F}}{l''}$$

$$\therefore l' l'' = \text{F}^2.$$

But $l' = f - \text{F}$ and $l'' = f' - \text{F}$.

On substituting these values in the above equation, we have—

$$ff' = Ff' + Ff.$$

On dividing by Fff', we have—

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}$$

It will be observed that the distances f and f' are reckoned from the first and second principal planes respectively, and not from the surfaces of the lens.

In fig. 45 $AK'K''A'$ is a secondary axis. The angle $AK'B$ is equal to the angle $A'K''B'$, and the angle $K'AB$ is equal to the angle $K''A'B'$, from which it follows that the size of the object is to that of the image as the first conjugate focal distance (f) is to the second (f'). If the object is in front of the image at twice the focal length, then the image is found at the same distance behind the lens, and object and image are of the same size. As the object approaches the lens, then the image recedes from it and increases in size, and

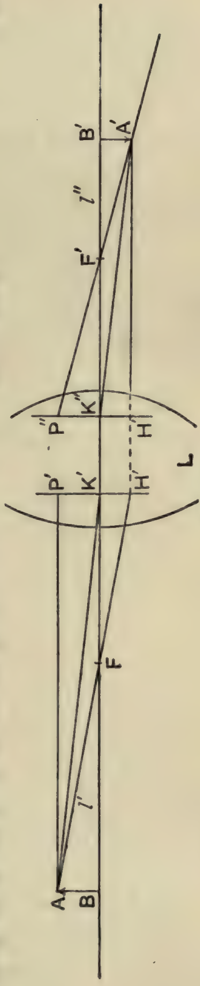


Fig. 45

when the object is placed at the principal focus the distance of the image from the lens is in the limit infinite. In other words, the light leaving the lens is in parallel pencils. When the object is at a greater distance from the lens than twice its focal length, then the image is smaller than the object. Lastly, if the object be nearer the lens than the principal focus, the rays of light after refraction do not converge, but continue to diverge as if coming from a virtual image on the same side of the lens. So long as the image and object are on opposite sides of the lens their distances are both positive.

The Biconcave Lens.—In this lens the optic centre and nodal points are found by a construc-

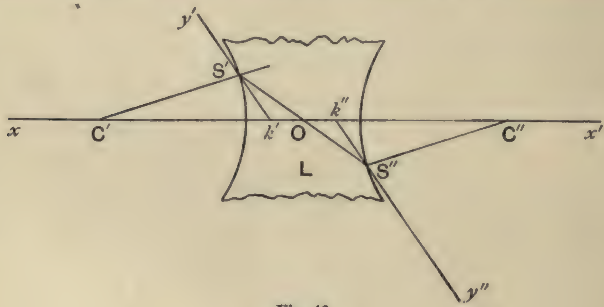


Fig. 46

tion similar to that already employed for the convex lens.

Let L (fig. 46) be a biconcave lens, of which $x x'$ is the principal axis. Let C′ be the centre of curvature of the first surface and C″ that of the second.

Draw the parallel radii C′S′ and C″S″. Join S′S″, cutting the principal axis at O: this is the optical centre. If O be the source of light the rays OS′ and OS″ take, after refraction, the directions S′y' and S″y'' respectively, and these are parallel to each other. Moreover, S′y', if prolonged backwards, intersects the principal axis at k' and S″y'' at k''. These points are therefore the nodal points, and if the medium on both surfaces is atmospheric air they are also the principal points.

By reasoning similar to that employed in the case of the biconvex lens, we have—

$$a = \frac{e f_1''}{f_1'' + f_2'}$$

$$b = \frac{e f_2'}{f_1'' + f_2'}$$

$$h' = \frac{e f_1'}{f_1'' + f_2' - e},$$

$$h'' = \frac{e f_2''}{f_1'' + f_2' - e}.$$

In applying these formulæ it is to be remembered that f_1', f_1'', f_2', and f_2'' are all negative, since they are situated on the same side of the surface as that from which the light comes. Hence a, b, h', and h'' are all positive, for in each case both numerator and denominator are negative, which means that the optical centre and the nodal and principal points are all situated in the interior of the lens.

For the principal focal length we have—

$$F = \frac{f_1' f_2'}{f_1'' + f_2' - e}.$$

Here, as f_1', f_1'', and f_2' are all negative, we have a positive numerator and a negative denominator, which gives a negative value to F. This means that the principal focal point is on the same side of the lens as the incident parallel light.

Moreover, by analogy (see p. 99) we have—

$$\frac{1}{f} + \frac{1}{f'} = -\frac{1}{F}.$$

Thick plano-concave or convex glasses are not used in ophthalmology, nor is the thick meniscus. Therefore it is not considered necessary to treat of them here.

APPENDIX

THE SPECTROMETER

This instrument (see fig. 47) consists of a circular horizontal metallic table accurately graduated into degrees.

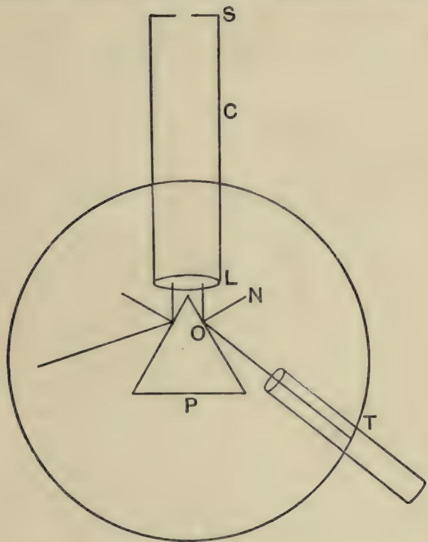


Fig. 47

To it is rigidly attached a tube C parallel to the plane of the table, and directed towards its centre. At one end of the tube is a narrow slit S, and at the other a

lens L, whose focal length is exactly the length of the tube. This part of the apparatus is called the collimator. Monochromatic light, such as the sodium flame, is placed opposite to the slit, which thus acts as a luminous source, and the light, after refraction at the lens L, emerges parallel to the axis of the collimator. The prism P, whose angle is to be measured, is placed near the centre of the table, with its apex pointing towards the collimator. From the diagram it is apparent that the light coming from the collimator is incident on one side of the prism at the point O, and is reflected at an angle equal to that of the incident angle. A telescope T, which moves in a circle having its centre exactly in a vertical line through the centre of the table, receives the reflected pencil, and the observer, by using the focussing screw, can obtain an image of the slit S. Inspection of the diagram shows that a similar pencil is reflected from the other side of the prism, and it is therefore possible, by moving the telescope round to the other side, again to obtain an image of the slit S.

It can be shown geometrically that the angle through which the telescope has been turned between the two positions in which an image of the slit is obtained, is equal to twice the apex angle of the prism. Let O P and O′ P′ (fig. 48) represent a pencil of parallel light incident on the apex of the prism D A E. A portion of the light is reflected in the direction P R, and another in the direction P′ R′. If P R and P′ R′ are produced backwards they meet at the point C; it is required to show that the angle R C R′ is twice the angle P A P′. Join A C and produce it to G. If A C be not parallel to O P or O′ P′, draw A F parallel to these lines.

From the law of reflection it follows that the angles RPD, OPA, and APC are all equal to each other. Simi-

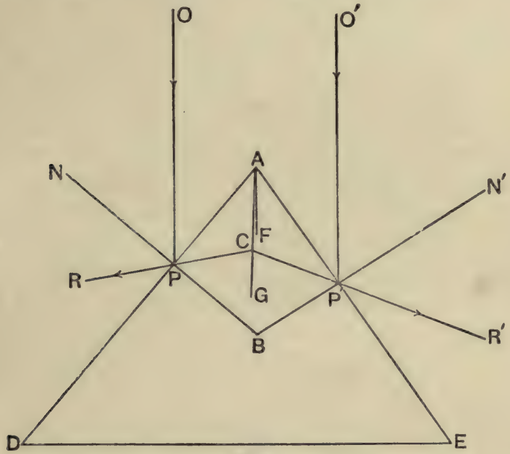


Fig. 48

larly the angles R′P′E, O′P′A, and AP′C are equal to each other. Again—

$$\angle PAP' = \angle PAF + \angle P'AF = \angle APC + \angle AP'C,$$

and—

$$\angle PAC + \angle P'AC = \angle PAF + \angle P'AF.$$

Again—

$$\angle RCR' = \angle RCG + \angle R'CG,$$

and—

$$\angle RCG = \angle PAC + \angle APC,$$

and—

$$\angle R'CG = \angle P'AC + \angle AP'C.$$

Therefore—

$$\angle RCR' = \angle PAC + \angle P'AC + \angle APC + \angle AP'C = 2\,PAP'.$$

In determining the position of minimum deviation

it is important that the prism be placed on the table of the spectrometer, so that its faces are at right angles to the plane of the table. In measuring the angle of minimum deviation the observer, before placing the prism on the instrument, first views the slit through the telescope direct. The collimator and telescope are then in line. The prism is now placed on the stand, with the line bisecting its apex angle as nearly as possible at right angles to the direction of the telescope. Light coming from the collimator, incident on the surface of the prism, in passing through the prism is deviated towards its base. It therefore no longer enters the telescope, which must be turned round before the observer again sees the image of the slit. Let us suppose that by moving it to the left the image is clearly seen. The observer brings the thread of the telescope to the centre of the image. On rotating the table in one direction the image will be seen to move to the right. If it be followed, a position is obtained at which the movement towards the right stops, and is succeeded by a movement to the left. That point marks the position of minimum deviation, for no matter in which direction the prism is rotated the deviation is increased. The angular distance, as read on the scale with a vernier, between the initial position, that in which the slit is viewed directly with the telescope and this latter point, is the angle of minimum deviation.

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